C2 – LIKE AND P2 – LIKE FINSLER SPACES

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Abstract
In the present communication certain properties of C2- like Finsler space have been studied. Necessary and sufficient conditions have been derived under which a C2-like Finsler space is a Landsberg space along with that a C2- like Finsler space is a P^* - Finsler space and a C2- like Finsler space is P2- like. In the last section of this communication studies have been carried out in a "Semi P2 - like" Finsler space. The necessary and sufficient conditions under which a semi P2 - like Finsler space is a Landsberg space, a non- Landsberg P^*- Finsler space is semi P2- like, a non- Lendsberg P2- like Finsler space is semi P2 like, a semi P2 like Finsler Space is P- symmetric have been derived.

1. Introduction

Matsumoto and Numata [6] have introduced the notation of C2 – like Finsler space. The (v) hν - curvature tensor of a C2-like Finsler space can be introduced in two different forms. With the help of these forms we can define semi-P2- like and P_λ- Finsler spaces. A generalization of P* [3] and semi-P2-like Finsler space is a P_λ-Finsler space F_n (n>2). The present paper has been divided into three sections of which are the first section is introductory. In the second section certain properties of C2-like Finsler spaces have been studied. Necessary and sufficient conditions have been derived in this section under which a C2-like Finsler space is a Landsberg space along with that a C2 - like Finsler space is a P*- Finsler space and a C2-like Finsler space is a P2- like and in this continuation have also established that if the hν-curvature tensor of a Finsler space is written in the form

$$P_{hijk} = C_j C_k E_{ih} + C_j M_{khi} + C_k M_{jhi} ,$$

where $E_{ih} = \frac{1}{c^2}(C_{i|h} - C_{h|i} + C_h Q_i \; C_i Q_h)$,

$Q_i = \frac{1}{c^2}(2C^i \; C_{1|i} + C^4 \; C_i)$,

and $M_{khi} = \frac{1}{c^2}(C_{k|h} C_i - C_{k|i} \; C_h)$,

then its (v) hν- torsion tensor can be written in the form

$$P_{ijk} = G_i C_j C_k + G_j C_k C_i + G_k C_i C_j - \phi \; C_i \; C_j \; C_k$$

where $C_i = \frac{1}{C^2} P_i$ and $\phi = \frac{2}{C^2} \; C_i \; C^i$.

In the third section of this communication studies have been carried out in a "Semi P2-like" Finsler spaces. The necessary and sufficient conditions under which a semi P2-like Finsler space is a Landsberg space, a non-Landsberg P*- Finsler space is semi P2-like, a non – Landsberg P2-like Finsler space is semi P2-like, a semi P2-like Finsler space is P-symmetric have been derived.

Let F_n be an n-dimensional Finsler space equipped with the fundamental function F($x, \dot{x}$). The (v) hv- torsion tensor P^i_{jk} and v-curvature tensor S_{hijk} are respectively given by the following relations

(1.1) $P^i_{hkm} = \frac{\partial \Gamma^{*i}_{hk}}{\partial x^m} - C^i_{hm|k} - C^i_{hr} P^r_{km}$,

(1.2) $S^i_{hkm} = C^i_{rk} C^r_{hm} - C^i_{rm} C^r_{hk}$.

and (1.3) (a) $P_{hijk} = g_{im} P^m_{hjk}$, (b) $S_{hijk} = g_{im} \; S^m_{hjk}$.

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We now propose to introduce the special Finsler spaces which shall be defined with the help spacial forms of curvature and torsion tensors.

In general $g_{ij(k)}$ = -2$C_{ijk|o}$ in a Finsler space where (k) stands for Berwald's process of covariant differentiation and suffix o denotes the transvection with respect to $\dot{x}^i$. A Landsberg space is characterized by the condition $g_{ij(k)}$= 0 .such a space is also characterized by the condition

(1.4) $P_{ijk} = C_{ijk|o} = 0$.

The notion of P*- Finsler space has been introduced by Izumi [3]. A Finsler space F_n (n>2) with the non-zero length C of the torsion vector C^i is called a P*-Finsler space, if the (v) hv- torsion tensor of the space is written in the form

(1.5) $P_{ijk} = \lambda C_{ijk}$

where $\lambda(x, \dot{x})$ is a scalar function given by

(1.6) (a) $\lambda = \frac{1}{C^2} P_i C^i$, (b) $C^i = g^{ij} C_j$,

(c) $C_i = C^j_{ij}$, (d) $P_i = P^j_{ij} = C_{i|o}$,

(e) $C^2 = g^{ij} C_i C_j$.

The P-symmetric [6] Finsler space is a special Finsler space which is defined with the special form of hν-curvature tensor P_{hijk}. Such a space is characterized by the relation

(1.7) $P_{hijk} - P_{hikj} = 0$.

A non- Riemannian Finsler space F_n of dimension n (n>2) is called P2-like [7], if there exists a covariant vector field λ_i such that the hv-curvature tensor P_{ijkl} of F_n is written in the form

(1.8) $P_{ijkl} = \lambda_i C_{jkl} - \lambda_j C_{ikl}$.

In such a Finsler space the hν- curvature tensor P_{hijk} of F_n vanishes, or the curvature tensor S_{hijk} of F_n vanishes.

A Finsler space F_n(n ≥2) with $C^2 \neq 0$ is called C2- like [6], if the (v) hv-torsion tensor C_{ijk} is written in the form

(1.9) $C_{ijk} = \frac{1}{C^2} C_i \; C_j C_k$.

It follows from (1.9) that a non-Riemannian Finsler space F_n(n ≥2) is C2- like if and only if C_{ijk} is written in the form

(1.10) $C_{ijk} = L_i C_j C_k + L_j C_k C_i + L_k C_i C_j$,

where $L_i = (\frac{1}{3C^2} C_i)$ are the components of a covariant vector field.

2. Properties of C2-Like Finsler Spaces:

The h-covariant differentiation of relation (1.9) with respect to x^i and thereafter transvection with respect $\dot{x}^i$, gives

(2.1) $P_{ijk} = G_i C_j C_k + G_j C_k C_i + G_k C_i C_j - \phi C_i C_j C_k$

where, we have written

$G_i = \frac{1}{C^2} P_i$, $\emptyset = \frac{2}{C^2} G_i C^i$ and P_i is defined by (1.6d).

The relation P_{ijk} = 0 directly gives P_i = 0. On the other hand, the condition P_i= 0, in view of (2.1) yields P_{ijk}= 0. Thus, we can state:

Theorem(2.1)
The necessary and sufficient condition in order that a C2-like Finsler space be a Landsberg space is given by P_i=0.

In view of (1.5), a necessary condition for a Finsler space to be P*- Finsler space is that $P_i = \lambda C_i$. But this condition is not sufficient. However, it has been observed that if F_n is C2-like than by virtue of (2.1), $P_i = \lambda C_i$ is a sufficient condition in order that the space be P*. Hence, we have

Theorem (2.2)
The necessary and sufficient condition in order that a C2- like Finsler space F_n(n>2) be a P*- Finsler space if and only if $P_i = \lambda C_i$ where λ is given by (1.6a).

On substituting from relation (1.9) and (2.1) into (1.3a) we get the following form of hv- curvature tensor

(2.2) $P_{hijk} = C_j C_k E_{ih} + C_j M_{khi} + C_k M_{jhi}$,

where (2.3) $E_{ih} = \frac{1}{C^2}(C_{i|h} - C_{h|i} + C_h Q_i - C_i Q_h)$,

(2.4) $Q_i = \frac{1}{C^2}(2C^i\, C_{1|i} + C^4 G_i)$,

(2.5) $M_{khi} = \frac{1}{C^2}(C_{k|h} C_i - C_{k|i} C_h)$.

After contracting (2.2) with respect to $\dot{x}^h$ and making use of equation (1.1), (2.3), (2.4) and (2.5) thereafter, we get the relation (2.1) and with the help of this observation, we can state that:

Theorem(2.3)
If the hv-curvature tensor of a Finsler space is written in the form (2.1), then its (v) hv-torsion tensor is written in the form (2.1).

The following corollary can be easily deduced from (2.2).

Corollary(2.1)
The hv-curvature tensor of a C2-like Finsler space F_n satisfies the identity $P_{hijk} = P_{hikj}$, that is a C2 – like Finsler space is P-symmetric.

Keeping in mind the relations (2.2), (2.3), (2.4) and (2.5), we have

(2.6) $P_{hijk} = K_h C_{ijk} - K_i C_{hjk} + M_{hijk}$,

where $K_h = -Q_h$

and $M_{hijk} = \frac{1}{C^2}(C_{i|h} - C_{h|i}) C_j C_k + C_j M_{khi} + C_k M_{jhi}$.

With the help of relations (1.7a) and (2.6) we can therefore state:

Theorem (2.4)
The necessary and sufficient condition in order that a C2-like Finsler space F_n is P2-like is that there should exists a covariant vector field U_i such that

$$M_{hijk} = U_h C_{ijk} - U_i C_{hjk} \ .$$

3. Semi P2-Like Finsler Spaces

We now consider an n-dimensional Finsler space F_n with the (v) hv- torsion tensor P_{ijk} of a special form to be given by

(3.1) $P_{ijk} = B_i C_j C_k + B_j C_k C_i + B_k C_i C_j$, where B_i is an indicatory vector field positively homogeneous of degree two in its directional arguments. From (3.1), it can easily be obtained that the vector B_i can be given by

(3.2) $B_i = \frac{1}{C^2}(P_i - \frac{2}{3C^2} P_k C^k C_i)$,

for $C^2 \neq 0$. Such a situation has arrived in the case of C2-likeness. Therefore, we given the following definition along with the adjoining remark.

Definition (3.1)

A non- Riemannian Finsler space F_n(n ≥ 2) is called semi P2-like, if the (v) hν-torsion tensor P_{ijk} of F_n is written in the form (3.1).

Remark (3.1)

In a Riemannian space , we have C_{ijk}= 0, which gives C_i=0 and C^2=0 in view of (1.6c) and (1.6e). Conversely, it C^2=0, that is C_i=0, then Dickey's theorem
shows that F_n is necessarily Riemannian. Thus we conclude that a Finsler Space
F_n is Riemannian if and only if C^2 = 0.

All these observation clearly indicate that in a semi-P2-like Finsler space, $C^2 \neq 0$. In two dimensional Finsler space F_2, we can easily get the following [8]

(3.3) (a) $C_{ijk} = \frac{J}{L} m_i m_j m_k$,

(b) $C_i = \frac{J}{L} m_i$, (c) $P_{ijk} = \frac{J_{|0}}{L} m_i m_j m_k$,

where m_i is a unit vector orthogonal to supporting element and J is the principal scalar. The relation (3.3b) and (3.3c) show that in every two dimensional Finsler space P_{ijk} can be expressed in the form (3.1) .

Therefore ,we can state:

Corollary (3.1)

A non-Riemannian two dimensional Finsler space F_2 is semi-P2-like.

with the help of (3.2) in to (3.1), we get the relation (2.1). and also (2.1) can always be expressed in the form (3.1). Therefore, we observe that if a C2-like Finsler space is semi-P2-like then such a semi-P2-like Finsler space is characterized by (2.1).

The condition P_i=0 in view of (3.1) and (3.2) gives P_{ijk}=0. Conversely, P_i=0 is the necessary condition for a Finsler space to be a Landsberg space. Therefore, we can state:

Theorem (3.1)

The necessary and sufficient condition in order that a semi-P2-like Finsler space be a Landsberg space if and only if P_i=0.

Certain examples are there of C2-like Finsler spaces, few of them are given below.

We now make the assumption that a semi-P2-like Finsler space F_n is a P*-Finsler space, then in view of the relation (1.5) and (3.1), we get

$$C_{ijk} = \frac{1}{\lambda} (B_i C_j C_k + B_j C_k C_i + B_k C_i C_j) \ .$$

Consequently, (1.9) tells that the space under consideration is C2-like provided λ≠0.

on the other hand λ=0will tell that the space under consideration is a Landsberg Space. Therefore, we can state:

Theorem (3.2)

The necessary and sufficient condition in order that a non- Landsberg P*-Finsler space be semi-P2-like is that it is a C2-like Finsler space.

We now consider a P2-like Finsler space. Contracting (1.a) with respect to $\dot{x}^i$, we get

$$P_{ijk} = \lambda_0 C_{ijk} ,$$

which shows that a P2-like Finsler space is a P*-Finsler space. Therefore, in view of theorem (3.2), we can state

Theorem (3.3)

A non-Landsberg P2-like Finsler space is semi-P2-like if and only if it is C2-like.

In view of the relation (3.1), (3.2) and $P_i = \mu C_i$ (assume), we get

$$(3.4)\quad P_{ijk} = \frac{\mu}{C^2} C_i C_j C_k .$$

Conversely, the relation (3.4) gives $P_i = \mu C_i$, which yields

Theoerm (3.4)

The (v) hv-torsion tensor of a semi-P2-like Finsler space is given by (3.4) if and only if $P_i = \mu C_i$.

In order to find the form of hv-curvature tensor P_{hijk} of semi-P2-like Finsler space, we have from (1.3) and (3.1)

$$(3.5)\quad P_{hijk} = \Theta_{(hi)} [B_{i|h} C_j C_k + C_j Q_{ikh} + C_k Q_{ijh} + C_i L_{jkh} + L_{ik} B_{jh} + C_i C_k F_{jh}]$$

where (3.6) (a) $Q_{ikh} = B_i C_{k|h} + C_i B_{k|h}$,

(b) $L_{jkh} = B_j C_{k|h} + B_k C_{j|h}$,

(c) $L_{ik} = C_i B_k + B_i C_k$,

(d) $B_{ij} = C_r C^r_{ij}$,

(e) $F_{ij} = B_k C^k_{ij}$,

We now consider a semi-P2-like Finsler space which is P- symmetric.

Keeping in mind (1.7) and (3.5), we get

$$(3.7)\quad \Theta_{(hi)} = [L_{ik} B_{jh} - L_{ij} B_{kh} + C_i C_k F_{jh} - C_i C_j F_{kh}] = 0 ,$$

Therefore, we can state:

Theorem (3.5)

The necessary and sufficient condition in order that a semi-P2-like Finsler space be P-symmetric is given by (3.7).

In the last, we consider that a semi-P2-like Finsler space admits a concurrent vector field X_i [5]. Then $X^i_{|j} = -\delta^i_j$ and $X^i_{|j} = 0$, which gives

$$(3.8)\quad P_{ijk} X^i = 0 , \quad C_{ijk} X^i = 0 .$$

From second relation of (3.8), we get $C_i X^i = 0$. Consequently, (3.1) with the help of these relations yields

$$B_i X^i C_j C_k = 0.$$

Contraction of the above relation with g^{jk} will give $B_i X^i = 0$. Therefore, we can state:

Theorem (3.6)

If a concurrent vector field X_i is admitted by a semi-P2-like Finsler space, then the vector B_i is orthogonal to X^i.

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