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f-REGULAR f-DERIVATIONS ON p-SEMI SIMPLE BCIK-ALGEBRAS

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Abstract
Introduced BCIK – algebra and its properties, and also we introduce the notion of derivation of a BCIK-algebra and investigate some related properties. Using the idea of regular f-derivation of a BCIK-algebra and investigate related properties. In this paper, the notion of left-right (resp., right-left) f-derivation of a BCIK-algebra is introduced, and some related properties are investigated. Using regular f-derivation, we give characterizations of a regular f-derivation on p -semi simple BCIK-algebra.

Keywords: BCIK-algebra, p-semi simple, f-derivations, f-regular.

1. Introduction

In 1966, Y. Imai and K. Iseki [1, 2] defined BCK – algebra in this notion originated from two different sources: one of them is based on the set theory the other is form the classical and non – classical propositional calculi. In [3]. Y.B. Jun and X.L. Xin applied the notion of derivation in ring and near – ring theory to BCI – algebras, and they also introduced a new concept called a derivation in BCI–algebras and its properties. In 2021 [4], S Rethina Kumar introduce combination BCK–algebra and BCI–algebra to define BCIK–algebra and its properties and also using Lattices theory to derived the some basic definitions, and they also the idea introduced a regular f-derivation in BCIK-algebras. We give the Characterizations f-derivation p-semi simple algebra and its properties. After the work of Jun and Xin (2004) [3], many research articles have appeared on the derivations of BCI-algebras In different aspects as follows: In 2021[5], S Rehina Kumar have given the notion of t-derivation of BCIK-algebras and studied p-semi simple BCIK—algebras by using the idea of regular t-derivation in BCIK-algebras have extended the results of BCIK-algebra in the same paper they defined and studied the notion of left derivation of BCIK-algebra and investigated some properties of left derivation in p-semi simple BCIK-algebras. In 2009 [6], Ozturk and Ceven have defined the notion of derivation and generalized derivation determined by a derivation for a complicated subtraction algebra and discussed some related properties. Also, in 2009 [6], Ozturk et al. have introduced the notion of generalized derivation in BCI-algebras and established some results. Further, they have given the idea of torsion free BCI-algebra and explored some properties. In 2010 [7], Al-Shehri has applied the notion of left-right (resp., right-left) derivation in BCI-algebra in BCI-algebra and obtained some of its properties. In 2011[19], IIbira et al, have studied the notion of left-right (resp., right-left) symmetric bi derivation in BCI-algebras.

Motivated by a lot work done on f-derivations of BCIK-algebra and on derivations of other related abstract algebraic structures, in this paper we introduce the notion of regular f-derivation p-semi simple BCIK-algebras and obtain some of its related properties.

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2. Preliminaries

Definition 2.1. [5] BCIK algebra

Let X be a non-empty set with a binary operation * and a constant 0. Then (X, * ,0) is called a BCIK Algebra, if it satisfies the following axioms for all x, y, z Є X:

(BCIK-1) x*y = 0, y*x = 0, z*x = 0 this imply that x = y = z.

(BCIK-2) ((x*y) * (y*z)) * (z*x) = 0.

(BCIK-3) (x*(x*y)) * y = 0.

(BCIK-4) x*x = 0, y*y = 0, z*z = 0.

(BCIK-5) 0*x =0, 0*y = 0, 0*z = 0.

For all x, y, z Є X. An inequality ≤ is a partially ordered set on X can be defined x ≤ y if and only if

(x*y) * (y*z) = 0.

Properties 2.2. [5] I any BCIK – Algebra X, the following properties hold for all x, y, z Є X:

(1) 0 Є X.
(2) x*0 = x.
(3) x*0 = 0 implies x = 0.
(4) 0*(x*y) = (0*x) * (0*y).
(5) X*y = 0 implies x = y.
(6) X*(0*y) = y*(0*x).
(7) 0*(0*x) = x.
(8) x*y Є X and x Є X imply y Є X.
(9) (x*y) * z = (x*z) * y
(10) x*(x*(x*y)) = x*y.
(11) (x*y) *(y*z) = x*y.
(12) 0 ≤ x ≤ y for all x, y Є X.
(13) x ≤ y implies x*z ≤ y*z and z*y ≤ z*x.
(14) x*y ≤ x.
(15) x*y ≤ z ⇔ x*z ≤ y for all x, y, z Є X
(16) x*a = x*b implies a = b where a and b are any natural numbers (i. e)., a, b Є N
(17) a*x = b*x implies a = b.
(18) a*(a*x) = x.

Definition 2.3. [4, 5, 10], Let X be a BCIK – algebra. Then, for all x, y, z Є X:

(1) X is called a positive implicative BCIK – algebra if (x*y) * z = (x*z) * (y*z).
(2) X is called an implicative BCIK – algebra if x*(y*x) = x.
(3) X is called a commutative BCIK – algebra if x*(x*y) = y*(y*x).
(4) X is called bounded BCIK – algebra, if there exists the greatest element 1 of X, and for any x Є X, 1*x is denoted by GG_x,
(5) X is called involutory BCIK – algebra, if for all x Є X, GG_x = x.

Definition 2.4. [5] Let X be a bounded BCIK-algebra. Then for all x, y Є X:

(1) G1 = 0 and G0 = 1,
(2) $GG_x \leq x$ that $GG_x = G(G_x)$,
(3) $G_x * G_y \leq y*x$,
(4) $y \leq x$ implies $G_x \leq G_y$,
(5) $G_{x*y} = G_{y*x}$
(6) $GGG_x = G_x$.

Theorem 2.5. [5] Let X be a bounded BCIK-algebra. Then for any x, y Є X, the following hold:

(1) X is involutory,
(2) $x*y = G_y * G_x$,
(3) $x*G_y = y * G_x$,
(4) $x \leq G_y$ implies $y \leq G_x$.

Theorem 2.6. [5] Every implicative BCIK-algebra is a commutative and positive implicative BCIK-algebra.

Definition 2.7. [4,5] Let X be a BCIK-algebra. Then:

(1) X is said to have bounded commutative, if for any x, y Є X, the set A(x,y) = {t Є X : $t*x \leq y$} has the greatest element which is denoted by x o y,
(2) (X, *, ≤) is called a BCIK-lattices, if (X, ≤) is a lattice, where ≤ is the partial BCIK-order on X, which has been introduced in Definition 2.1.

Definition 2.8. [5] Let X be a BCIK-algebra with bounded commutative. Then for all x, y, z Є X:

(1) $y \leq x \text{ o } (y*x)$,
(2) $(x \text{ o } z) * (y \text{ o } z) \leq x*y$,
(3) $(x*y) * z = x*(y \text{ o } z)$,
(4) If $x \leq y$, then $x \text{ o } z \leq y \text{ o } z$,
(5) $z*x \leq y \Leftrightarrow z \leq x \text{ o } y$.

Theorem 2.9. [4,5] Let X be a BCIK-algebra with condition bounded commutative. Then, for all x, y, z Є X, the following are equivalent:

(1) X is a positive implicative,
(2) $x \leq y$ implies x o y = y,
(3) x o x = x,
(4) (x o y) * z = (x*z) o (y*z),
(5) x o y = x o (y*x).

Theorem 2.10. [4,5] Let X be a BCIK-algebra.

(1) If X is a finite positive implicative BCIK-algebra with bounded and commutative the (X, ≤) is a distributive lattice,
(2) If X is a BCIK-algebra with bounded and commutative, then X is positive implicative if and only if (X, ≤) is an upper semi lattice with $x \vee y = x \text{ o } y$, for any x, y Є X,
(3) If X is bounded commutative BCIK-algebra, then BCIK-lattice (X, ≤) is a distributive lattice, where $x \wedge y = y*(y*x)$ and $x \vee y = G(G_x \wedge G_y)$.

Theorem 2.11. [4,5] Let X be an involutory BCIK-algebra, Then the following are equivalent:
(1) (X, ≤) is a lower semi lattice,
(2) (X, ≤) is an upper semi lattice,
(3) (X, ≤) is a lattice.

Theorem 2.12. [5] Let X be a bounded BCIK-algebra. Then:
(1) every commutative BCIK-algebra is an involutory BCIK-algebra.
(2) Any implicative BCIK-algebra is a Boolean lattice (a complemented distributive lattice).

Theorem 2.13. [5, 11] Let X be a BCK-algebra, Then, for all x, y, z Є X, the following are equivalent:
(1) X is commutative,
(2) x*y = x*(y*(y*x)),
(3) x*(x*y) = y*(y*(x*(x*y))),
(4) x≤ y implies x = y*(y*x).

3. Regular Left derivation p-semi simple BCIK-algebra

Definition 3.1. Let X be a p-semi simple BCIK-algebra. We define addition + as x + y = x*(0*y) for all x, y Є X. Then (X, +) be an abelian group with identity 0 and x –y = x*y. Conversely, let (X, +) be an abelian group with identity 0 and let x – y =x*y. Then X is a p-semi simple BCIK-algebra and x + y = x*(0*y),
for all x, y Є X (see [6]). We denote x y = y * (y * x), 0 * (0 * x) = a_x and
L_p(X) = {a Є X / x * a = 0 implies x = a, for all x Є X}.
For any x Є X. V(a) = {a Є X / x * a = 0} is called the branch of X with respect to a. We have x * y Є V (a * b), whenever x Є V(a) and y Є V(b), for all x, y Є X and all a, b Є L_p(X), for 0 * (0 * a_x) = a_x which implies that a_x * y Є L_p(X) for all y Є X. It is clear that G(X) ⊂ L_p(X) and x * (x * a) = a and
a * x Є L_p(X), for all a Є L_p(X) and all x Є X.

Definition 3.2. ([5]) Let X be a BCIK-algebra. By a (l, r)-derivation of X, we mean a self d of X satisfying the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y)) \text{ for all } x, y Є X.$$

If X satisfies the identity

$$d(x * y) = (x * d(y)) \wedge (d(x) * y) \text{ for all } x, y Є X,$$

then we say that d is a (r, l)-derivation of X
Moreover, if d is both a (r, l)-derivation and (r, l)-derivation of X, we say that d is a derivation of X.

Definition 3.3. ([5]) A self-map d of a BCIK-algebra X is said to be regular if d (0) = 0.
Definition 3.4. ([5]) Let d be a self-map of a BCIK-algebra X. An ideal A of X is said to be d-invariant, if d(A) = A.
In this section, we define the left derivations

Definition 3.5. Let X be a BCIK-algebra By a left derivation of X, we mean a self-map D of X satisfying

$$D(x * y) = (x * D(y)) \wedge (y * D(x)), \text{ for all } x, y Є X.$$

Example 3.6. Let X = {0,1,2} be a BCIK-algebra with Cayley table defined by

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a map D: X $\rightarrow$ X by

$$D(x) = \begin{cases} 2 \text{ if } x = 0,1 \\ 0 \text{ if } x = 2. \end{cases}$$

Then it is easily checked that D is a left derivation of X.

Proposition 3.7. Let D be a left derivation of a BCIK-algebra X. Then for all x, y Є X, we have

(1) x * D(x) = y * D(y).
(2) D(x) = $a_{D(x)\ x}$.
(3) D(x) = D(x) $\wedge$ x.
(4) D(x) Є L_p(X).

Proof.

(1) Let x, y Є X. Then

D(0) = D(x * x) = (x * D(x)) $\wedge$ (x * D(x)) = x * D(x).

Similarly, D(0) = y * D(y). So, D(x) = y * D(y).

2) Let x Є X. Then

D(x) = D(x * 0)
= (x * D(0)) $\wedge$ (0 * D(x))
= (0 * D(x)) * ((0 * D(x)) * (x * D(0)))
$\leq$ 0 * (0 * (x * D(x))))
= 0 * (0 * (x * (x * D(x))))
= 0 * (0 * (D(x) $\wedge$ x))
= $a_{D(x)\ x}$.

Thus D(x) $\leq$ $a_{D(x)\ x}$. But

$a_{D(x)\ x}$ = 0(0 * (D(x) $\wedge$ x)) $\leq$ D(x) $\wedge$ x $\leq$ D(x).

Therefore, D(x) = $a_{D(x)\ x}$.

(3) Let x Є X. Then using (2), we have

D(x) = $a_{D(x)\ x}$ $\leq$ D(x) $\wedge$ x .

But we know that D(x) $\wedge$ x $\leq$ D(x), and hence (3) holds.

(4) Since a_x Є L_p(X), for all x Є X, we get D(x) Є L_p(X) by (2).

Remark 3.8. Proposition 3.3(4) implies that D(X) is a subset of L_p(X).

Proposition 3.9. Let D be a left derivation of a BCIK-algebra X. Then for all x, y Є X, we have

(1) Y * (y * D(x)) = D(x).
(2) D(x) * y Є L_p(X).

Proposition 3.10. Let D be a left derivation of a BCIK-algebra of a BCIK-algebra X. Then

(1) D(0) Є L_p(X).

(2) D(x) = 0 + D(x), for all x Є X.
(3) D(x + y) = x + D(y), for all x, y Є $L_p(X)$.
(4) D(x) = x, for all x Є X if and only if D(0) = 0.
(5) D(x) Є G(X), for all x Є G(X).

Proof.

(1) Follows by Proposition 3.3(4).
(2) Let x Є X. From Proposition 3.3(4), we get D(x) = $a_{D(x)}$, so we have

D(x) = $a_{D(x)}$ = 0 * (0 * D(x)) = 0 + D(x).

(3) Let x, y Є $L_p(X)$. Then

D(x + y) = D(x * (0 * y))
= (x * D(0 * y)) ∧ ((0 * y) * D(x))
= ((0 * y) * D(x)) * (((0 * y) * D(x) * (x * D(0 * y)))
= x * D(0 * y)
= x * ((0 * D(y)) ∧ (y * D(0)))
= x * D(0 *y)
= x * (0 * D(y))
= x + D(y).

(4) Let D(0) = 0 and x Є X. Then

D(x) = D(x) ∧ x = x * (x * D(x)) = x * D(0) = x * 0 = x.
Conversely, let D(x) = x, for all x Є X. So it is clear that D(0) = 0.

(5) Let x Є G(x). Then 0 * = x and so

D(x) = D(0 * x)
= (0 * D(x)) ∧ (x * D(0))
= (x * D(0)) * ((x * D(0)) * (0 * D(x))
= 0 * D(x).
This give D(x) Є G(X).

Remark 3.11. Proposition 3.6(4) shows that a regular left derivation of a BCIK-algebra is the identity map. So we have the following:
Proposition 3.12. A regular left derivation of a BCIK-algebra is trivial.
Remark 3.13. Proposition 3.6(5) gives that D(x) Є G(X) $\subseteq$ $L_p(X)$.
Definition 3.14. An ideal A of a BCIK-algebra X is said to be D-invariant if D(A)$\subset$A.
Now, Proposition 3.8 helps to prove the following theorem.

Theorem 3.15. Let D be a left derivation of a BCIK-algebra X. Then D is regular if and only if ideal of X is D-invariant.
Proof.
Let D be a regular left derivation of a BCIK-algebra X. Then Proposition 3.8. gives that D(x) = x, for all
x Є X. Let y Є D(A), where A is an ideal of X. Then y = D(x) for some x Є A. Thus

Y * x = D(x) * x = x * x = 0 Є A.

Then y Є A and D(A) $\subset$A. Therefore, A is D-invariant.

Conversely, let every ideal of X be D-invariant. Then D({0}) $\subset$ {0} and hence D(0) and D is regular.

Finally, we give a characterization of a left derivation of a p-semi simple BCIK-algebra.

Proposition 3.16. Let D be a left derivation of a p-semi simple BCIK-algebra. Then the following hold for all x, y Є X:

(1) D(x * y) = x * D(y).
(2) D(x) * x = D(y) * Y.
(3) D(x) * x = y * D(y).

Proof.

(1) Let x, y Є X. Then
D(x * y) = (x * D(y)) ∧ ∧ (y * D(x)) = x * D(y).

(2) We know that
(x * y) * (x * D(y)) ≤ D(y) * y and
(y * x) * (y * D(x)) ≤ D(x) * x.
This means that
((x * y) * (x * D(y))) * (D(y) * y) = 0, and
((y * x) * (y * D(x))) * (D(x) * x) = 0.
So
((x * y) * (x * D(y))) * (D(y) * y) = ((y * x) * (y * D(x))) * (D(x) * x). (I)
Using Proposition 3.3(1), we get,
(x * y) * D(x * y) = (y * x) * D(y * x). (II)
By (I), (II) yields
(x * y) * (x * D(y)) = (y * x) * (y * D(x)).
Since X is a p-semi simple BCIK-algebra. (I) implies that
D(x) * x = D(y) * y.

(3) We have, D(0) = x * D(x). From (2), we get D(0) * 0 = D(y) * y or D(0) = D(y) * y.
So D(x) * x = y * D(y).

Theorem 3.17. In a p-semi simple BCIK-algebra X a self-map D of X is left derivation if and only if and if it is derivation.

Proof.

Assume that D is a left derivation of a BCIK-algebra X. First, we show that D is a (r, l)-derivation of X. Then

D(x * y) = x * D(y)
= (D(x) * y) * ((D(x) * Y) * (x * D(y)))
= (x * D(y)) ∧ (D(x) * y).

Now, we show that D is a (r, l)-derivation of X. Then

D(x * Y) = x * D(y)
= (x * 0) * D(y)
= (x * (D(0) * D(0)) * D(y)
= (x * ((x * D(x)) * (D(y) * y))) * D(y)
= (x * ((x * D(y)) * (D(x) * y))) * D(y)
= (x * D(y) * ((x * D(y)) * (D(x) * Y))
= (D(x) * y) ∧ (x * D(y)).

Therefore, D is a derivation of X.

Conversely, let D be a derivation of X. So it is a (r, l)-derivation of X. Then

D(x * y) = (x * D(y)) ∧ (D(x) * y)

$= (D(x) * y) * ((D(x) * y) * (x * D(y)))$
$= x * D(y) = (y * D(x)) * ((y * D(x)) * (x * D(y)))$
$= (x * D(y)) \wedge (y * D(x))$.

Hence, D is a left derivation of X.

4. t-Derivations in BCIK-algebra /p-Semi simple BCIK-algebra

The following definitions introduce the notion of t-derivation for a BCIK-algebra.

Definition 4.1. Let X be a BCIK-algebra. Then for t Є X, we define a self-map $d_t : X \to X$ by $d_t(x) = x * t$ for all x Є X.

Definition 4.2. Let X be a BCIK-algebra. Then for any t Є X, a self-map $d_t : X \to X$ is called a left-rifht t-derivation or (l,r)-t-derivation of X if it satisfies the identity $d_t(x * Y) = (d_t(x) * y) \wedge (x * d_t(y))$ for all
x, y Є X.

Definition 4.3. Let X be a BCIK-algebra. Then for any t Є X, a self-map $d_t : X \to X$ is called a left-right t-derivation or (l, r)-t-derivation of X if it satisfies the identity $d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$ for all x, y Є X.

Moreover, if d_t is both a (l, r) and a (r, l)-t-derivation on X, we say that d_t is a t-derivation on X.

Example 4.4. Let X = {0,1,2} be a BCIK-algebra with the following Cayley table:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

For any t Є X, define a self-map $d_t : X \to X$ by $d_t(x) = x * t$ for all x Є X. Then it is easily checked that d_t is a t-derivation of X.

Proposition 4.5. Let d_t be a self-map of an associative BCIK-algebra X. Then d_t is a (l, r)-t-derivation of X.

Proof. Let X be an associative BCIK-algebra, then we have

$d_t(x * y) = (x * y)$
$= \{x * (y * t)\} * 0$
$= \{x * (y * t)\} * [\{x * (y * t)\} *\{x * (y *t)\}]$
$= \{x * (y * t)\} * [\{x * (y *t)\} * \{(x * y) * t\}]$
$= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * t) * y\}]$
$= ((x * t) * y) \wedge (x * (y * t))$
$= (d_t(x) * y) \wedge (x * d_t(y))$.

Proposition 4.6. Let d_t be a self-map of an associative BCIK-algebra X. Then, d_t is a (r, l)-t-derivation of X.

Proof. Let X be an associative BCIK-algebra, then we have

$d_t(x * y) = (x * y) * t$
$= \{(x * t) * y\} * 0$

$= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * t) * y)]$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * y) * t\}]$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{x * (y * t)\}]$
$= (x * (y * t)) \wedge ((x * t) * y)$
$= (x * d_t(y)) \wedge (d_t(x) * y)$

Combining Propositions 4.5 and 4.6, we get the following Theorem.

Theorem 4.7. Let d_t be a self-map of an associative BCIK-algebra X. Then, d_t is a t-derivation of x.
Definition 4.8. A self-map d_t of a BCIK-algebra X is said to be t-regular if $d_t(0) = 0$.
Example 4.9. Let X = {0, a, b} be a BCIK-algebra with the following Cayley table:

*	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(1) For any t Є X, define a self-map $d_t : X \to X$ by

$$d_t(x) = x * t = \begin{cases} b \text{ if } x = 0, a \\ 0 \text{ if } x = b \end{cases}$$

Then it is easily checked that d_t is (l, r) and (r, l)-t-derivations of X, which is not t-regular.
(2) For any t Є X, define a self-map $d'_t : X \to X$ by

$$d_t'(x) = x * t = \begin{cases} 0 \text{ if } x = 0, a \\ b \text{ if } x = b. \end{cases}$$

Then it is easily checked that d_t' is (l, r) and (r, l)-t-derivations of X, which is t-regular.

Proposition 4.10. Let d_t be a self-map of a BCIK-algebra X. Then
(1) If d_t is a (l, r)-t- derivation of x, then $d_t(x) = d_t(x) \wedge x$ for all x Є X.
(2) If d_t is a (r, l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all x Є X if and only if d_t is t-regular.

Proof.
(1) Let d_t be a (l, r)-t-derivation of X, then

$d_t(x) = d_t(x * 0)$
$= (d_t(x) * 0) \wedge (x * d_t(0))$
$= d_t(x) \wedge (x * d_t(0))$
$= \{x * d_t(0)\} * [\{x * d_t(0)\} * d_t(x)]$
$= \{x * d_t(0)\} * [\{x * d_t(x)\} * d_t(0)]$
$\leq x * \{x * d_t(x)\}$
$= d_t(x) \wedge x.$

But $d_t(x) \wedge x \leq d_t(x)$ is trivial so (1) holds.
(2) Let d_t be a (r, l)-t-derivation of X. If $d_t(x) = x \leq d_t(x)$ then

$d_t(0) = 0 \wedge d_t(0)$
$= d_t(0) * \{ d_t(0) * 0\}$
$= d_t(0) * d_t(0)$
$= 0$

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Thereby implying d_t is t-regular. Conversely, suppose that d_t is t-regular, that is $d_t(0) = 0$, then we have

$$\begin{aligned} d_t(0) &= d_t(x * 0) \\ &= (x * d_t(0)) \wedge (d_t(x) * 0) \\ &= (x * 0) \wedge d_t(x) \\ &= x \wedge d_t(x). \end{aligned}$$

The completes the proof.

Theorem 4.11. Let d_t be a (l, r)-t-derivation of a p-semi simple BCIK-algebra X. Then the following hold:

(1) $d_t(0) = d_t(x) * x$ for all $x \in X$.
(2) d_t is one-0ne.
(3) If there is an element $x \in X$ such that $d_t(x) = x$, then d_t is identity map.
(4) If $x \leq y$, then $d_t(x) \leq d_t(y)$ for all $x, y \in X$.

Proof.

(1) Let d_t be a (l, r)-t-derivation of a p-semi simple BCIK-algebra X. Then for all $x \in X$, we have $x * x = 0$ and so

$$\begin{aligned} d_t(0) &= d_t(x * x) \\ &= (d_t(x) * x) \wedge (x * d_t(x)) \\ &= \{x * d_t(x)\} * [\{x * d_t(x)\} * \{d_t(x) * x\}] \\ &= d_t(x) * x \end{aligned}$$

(2) Let $d_t(x) = d_t(y) \Rightarrow x * t = y * t$, then we have $x = y$ and so d_t is one-one.
(3) Let d_t be t-regular and $x \in X$. Then, $0 = d_t(0)$ so by the above part(1), we have $0 = d_t(x) * x$ and, we obtain $d_t(x) = x$ for all $x \in X$. Therefore, d_t is the identity map.
(4) It is trivial and follows from the above part (3).

Let $x \leq y$ implying $x * y = 0$. Now,

$$\begin{aligned} d_t(x) * d_t(y) &= (x * t) * (y * t) \\ &= x * y \\ &= 0. \end{aligned}$$

Therefore, $d_t(x) \leq d_t(y)$. This completes proof.

Definition 4.12. Let d_t be a t-derivation of a BCIK-algebra X. Then, d_t is said to be an isotone t-derivation if $x \leq y \Rightarrow d_t(x) \leq d_t(y)$ for all $x, y \in X$.

Example 4.13. In Example 4.9(2), d_t' is an isotone t-derivation, while in Example 4.9(1), d_t is not an isotone t-derivation.

Proposition 4.14. Let X be a BCIK-algebra and d_t be a t-derivation on X. Then for all $x, y \in X$, the following hold:

(1) If $d_t(x \wedge y) = d_t(x)\, d_t(x)\, d_t(x)$, then d_t is an isotone t-derivation
(2) If $d_t(x \wedge y) = d_t(x) * d_t(y)$, then d_t is an isotone t-derivation.

Proof.

(1) Let $d_t(x \wedge y) = d_t(x) \wedge d_t(x)$. If $x \leq y \Rightarrow x \wedge y = x$ for all $x, y \in X$. Therefore, we have

$$d_t(x) = d_t(x \wedge y)$$

$= d_t(x) \wedge d_t(y)$
$\leq d_t(y)$.

Henceforth $d_t(x) \leq d_t(y)$ which implies that d_t is an isotone t-derivation.

(2) Let $d_t(x * y) = d_t(x) * d_t(y)$. If $x \leq y \Rightarrow x * y = 0$ for all x, y Є X. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= d_t\{x * (x * y)\} \\ &= d_t(x) * d_t(x * y) \\ &= d_t(x) * \{ d_t(x) * d_t(y)\} \\ &\leq d_t(y). \end{aligned}$$

Thus, $d_t(x) \leq d_t(y)$. This completes the proof.

Theorem 4.15. Let d_t be a t-regular (r, l)-t-derivation of a BCIK-algebra X. Then, the following hold:

(1) $d_t(x) \leq x$ for all x Є X.
(2) $d_t(x) * y \leq x * d_t(y)$ for all x, y Є X.
(3) $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$ for all x, y Є X.
(4) $Ker(d_t) = \{ x \in X : d_t(x) = 0\}$ is a sub algebra of X.

Proof.

(1) For any x Є X,
we have $d_t(x) = d_t(x * 0) = (x * d_t(0)) \wedge (d_t(x) * 0) = (x * 0) \wedge (d_t(x) * 0) = x \wedge d_t(x) \leq x$.
(2) Since $d_t(x) \leq x$ for all x Є X, then $d_t(x) * y \leq x * y \leq x * d_t(y)$ and hence the proof follows.
(3) For any x, y Є X, we have

$$\begin{aligned} d_t(x * y) &= (x * d_t(y)) \wedge (d_t(x) * y) \\ &= \{d_t(x) * y\} * [\{d_t(x) * y\} * \{x * d_t(x)\}] \\ &= \{d_t(x) * y\} * 0 \\ &= d_t(x) * y \leq d_t(x) * d_t(x). \end{aligned}$$

(4) Let x, y Є ker $(d_t) \Rightarrow d_t(x) = 0 = d_t(y)$. From (3), we have $d_t(x * y) \leq d_t(x) * d_t(y) = 0 * 0 = 0$ implying $d_t(x * y) \leq 0$ and so $d_t(x * y) = 0$. Therefore, x * y Є ker (d_t). Consequently, ker(d_t) is a sub algebra of X. This completes the proof.

Definition 4.16. Let X be a BCIK-algebra and let d_t,d_t’ be two self-maps of X. Then we define
d_t o d_t’ : $X \rightarrow X$ by (d_t o d_t’)(x) = $d_t(d_t$’(x)) for all x Є X.

Example 4.17. Let X = {0, a, b} be a BCIK-algebra which is given in Example 4.4. Let d_t and d_t’ be two
self-maps on X as define in Example 4.9(1) and Example 4.9(2), respectively.

Now, define a self-map d_t o d_t’ : $X \rightarrow X$ by

$$(d_t \text{ o } d_t')(x) = \begin{cases} 0 \, if \;\; x = a, b \\ b \, if \;\; x = 0. \end{cases}$$

Then, it easily checked that (d_t o d_t’) (x) = $d_t(d_t$’(x)) for all x Є X.

Proposition 4.18. Let X be a p-semi simple BCIK-algebra X and let d_t, d_t’ be (l, r)-t-derivations of X. Then, d_t o d_t’ is also a (l, r)-t-derivation of X.

Proof. Let X be a p-semi simple BCIK-algebra. d_t and d_t' are (l, r)-t-derivations of X. Then for all x, y Є X, we get

$$(d_t \circ d_t')(x * y) = d_t(d_t'(x,y))$$
$$= d_t[(d_t'(x) * y) \wedge (x * d_t(y))]$$

$$= d_t[(x * d_t'(y)) * \{(x * d_t(y)) * (d_t'(x) * y)\}]$$
$$= d_t(d_t'(x) * y)$$
$$= \{x * d_t(d_t'(y))\} * [\{x * d_t(d_t'(y))\} * \{d_t(d_t'(x) * y)\}]$$

$$= \{d_t(d_t'(x) * y)\} \wedge \{x * d_t(d_t'(y))\}$$
$$= ((d_t \circ d_t')(x) * y) \wedge (x * (d_t \circ d_t')(y)).$$

Therefore, $(d_t \circ d_t')$ is a (l, r)-t-derivation of X.

Similarly, we can prove the following.

Proposition 4.19. Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be (r, l)-t-derivations of X. Then,
$d_t \circ d_t'$ is also a (r, l)-t-derivation of X.

Combining Propositions 3.18 and 3.19, we get the following.

Theorem 4.20. Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be t-derivations of X. Then, $d_t \circ d_t'$ is also a t-derivation of X.

Now, we prove the following theorem

Theorem 4.21. Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be t-derivations of X.
Then $d_t \circ d_t' = d_t' \circ d_t$.

Proof. Let X be a p-semi simple BCIK-algebra. d_t and d_t', t-derivations of X. Suppose d_t' is a (l, r)-t-derivation, then for all x, y Є X, we have

$$(d_t \circ d_t')(x * y) = d_t(d_t'(x * y))$$
$$= d_t[(d_t'(x) * y) \wedge (x * d_t(y))]$$
$$= d_t[(x * d_t'(y)) * \{(x * d_t(y)) * (d_t'(x) * y)\}]$$
$$= d_t(d_t'(x) * y)$$

As d_t is a (r, l)-t-derivation, then

$$= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y)$$
$$= d_t'(x) * d_t(y).$$

Again, if d_t is a (r, l)-t-derivation, then we have

$$(d_t \circ d_t')(x * y) = d_t'[d_t(x * y)]$$
$$= d_t'[(x * d_t(y)) \wedge (d_t(x) * y)]$$
$$= d_t'[x * d_t(y)]$$

But d_t' is a (l, r)-t-derivation, then

$$= (d_t'(x) * d_t(y)) \wedge (x * d_t'(d_t(y))$$
$$= d_t'(x) * d_t(y)$$

Therefore, we obtain

$$(d_t \circ d_t')(x * y) = (d_t' \circ d_t)(x * y).$$

By putting y = 0, we get

$$(d_t \circ d_t')(x) = (d_t' \circ d_t)(x) \text{ for all } x \in X.$$

Hence, $d_t \circ d_t' = d_t' \circ d_t$. This completes the proof.

Definition 4.22. Let X be a BCIK-algebra and let d_t,d_t' two self-maps of X. Then we define $d_t * d_t' : X \rightarrow X$ by $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all x Є X.

Example 4.23. Let X = {0, a, b} be a BCIK-algebra which is given in Example 3.4. let d_t and d_t' be two Self-maps on X as defined in Example 4.9 (1) and Example 4.10 (2), respectively.

Now, define a self-map $d_t * d_t' : X \rightarrow X$ by

$$(d_t * d_t')(x) = \begin{cases} 0\, if\ x = a, b \\ b\, if\ x = 0. \end{cases}$$

Then, it is easily checked that $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all x Є X.

Theorem 4.24. Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be t-derivations of X. Then $d_t * d_t' = d_t' * d_t$.

Proof. Let X be a p-semi simple BCIK-algebra. d_t and d_t', t-derivations of X.
Since d_t' is a (r, l)-t-derivation of X, then for all x, y Є X, we have

$$\begin{aligned}(d_t \circ d_t')(x * y) &= d_t(d_t'(x * y)) \\ &= d_t[(x * d_t'(y)) \wedge (d_t'(x) * y)] \\ &= d_t[(x * d_t'(y))]\end{aligned}$$

But d_t is a (l, r)-r-derivation, so

$$\begin{aligned}&= (d_t(x) * d_t'(y)) \wedge (x * d_t(d_t'(y)) \\ &= d_t(x) * d_t'(x).\end{aligned}$$

Again, if d_t' is a (l, r)-t-derivation of X, then for all x, y Є X, we have

$$\begin{aligned}(d_t \circ d_t')(x * y) &= d_t[d_t'(x * y)] \\ &= d_t[(d_t'(x) * y) \wedge (x * d_t'(y))] \\ &= d_t[(x * d_t'(y)) * \{(x * d_t'(y)) * (d_t'(x) * y)\}] \\ &= d_t(d_t'(x) * y).\end{aligned}$$

As d_t is a (r, l)-t-derivation, then

$$\begin{aligned}&= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y) \\ &= d_t'(x) * d_t(y).\end{aligned}$$

Henceforth, we conclude

$$d_t(x) * d_t'(y) = d_t'(x) * d_t(y)$$

By putting y =x, we get

$$d_t(x) * d_t'(x) = d_t'(x) * d_t(x)$$

$$(d_t * d_t')(x) = (d_t' * d_t)(x) \text{ for all } x \in X.$$

Hence $d_t * d_t' = d_t' * d_t$. This completes the proof.

5. f-derivation of BCIK-algebra

In what follows, let be an endomorphism of X unless otherwise specified.

Definition 5.1. Let X be a BCIK algebra. By a left f-derivation (briefly, (l, r)-f-derivation) of X, a self-map $d_f(x * y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$ for all x, y Є X is meant, where f is an endomorphism of

X. If d_f satisfies the identity $d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$ for all x, y Є X, then it is said that d_f is a right-left f-derivation (briefly, (r, l)-f-derivation) of X. Moreover, if d_f is both an (r, l)-f-derivation, it is said that d_f is an f-derivation.

Example 5.2. Let X = {0,1,2,3,4,5} be a BCIK-algebra with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Define a Map $d_f : X \to X$ by

$$d_f = \begin{cases} 2\,if \;\; x = 0,1, \\ 0\,otherwise, \end{cases}$$

and define and endomorphism f of X by

$$f(x) = \begin{cases} 2\,if \;\; x = 0,1, \\ 0\,otherwise, \end{cases}$$

That it is easily checked that d_f is both derivation and f-derivation of X.

Example 5.3. Let X be a BCIK-algebra as in Example 2.2. Define a map $d_f : X \to X$ by

$$d_f = \begin{cases} 2\,if \;\; x = 0,1, \\ 0\,otherwise, \end{cases}$$

Then it is easily checked that d_f is a derivation of X.

Define an endomorphism f of X by

f(x) = 0, for all x Є X.

Then d_f is not an f-derivation of X since

$$d_f(2 * 3) = d_f(0) = 2,$$

but

$$(d_f(2) * f(3)) \wedge (f(2) * d_f(3)) = (0 * 0) \wedge (0 * 0) = 0 \wedge 0 = 0,$$

And thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \wedge (f(2) * d_f(3))$.

Remark 5.4. From Example 5.3, we know that there is a derivation of X which is not an f-derivation X.
Example 2.5. Let X = {0,1,2,3,4,5} be a BCIK-algebra with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	1	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Define a map $d_f : X \rightarrow X$ by

$$d_f(x) = \begin{cases} 0 \text{ if } x = 0,1, \\ 2 \text{ if } x = 2,4, \\ 3 \text{ if } x = 3,5, \end{cases}$$

and define an endomorphism f of X by

$$f(x) = \begin{cases} 0 \text{ if } x = 0,1, \\ 2 \text{ if } x = 2,4, \\ 3 \text{ if } x = 3,5, \end{cases}$$

Then it is easily checked that d_f is both derivation and f-derivation of X.

Example 5.6. Let X be a BCIK-algebra as in Example 5.5. Define a map $d_f : X \rightarrow X$ by

$$d_f(x) = \begin{cases} 0 \text{ if } x = 0,1, \\ 2 \text{ if } x = 2,4, \\ 3 \text{ if } x = 3,5, \end{cases}$$

Then it is easily checked that d_f is a derivation of X.

Define an endomorphism f of X by

$f(0) = 0$, $f(1) = 1$, $f(2) = 3$ $f(3) = 2$, $f(4) = 5$, $f(5) = 4$.

Then d_f is not an f-derivation of X since

$$d_f(2 * 3) = d_f(3) = 3,$$

but

$$(d_f(2) * f(3)) \wedge (f(2) * d_f(3)) = (2 * 2) \wedge (3 * 3) = 0 \wedge 0 = 0,$$

And thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \wedge (f(2) * d_f(3))$.

Example 5.7. Let X be a BCIK-algebra as in Example 2.5. Define a map $d_f : X \rightarrow X$ by

$d_f(0) = 0$, $d_f(1) = 1$, $d_f(2) = 3$, $d_f(3) = 2$, $d_f(4) = 5$, $d_f(5) = 4$,

Then d_f is not a derivation of X since

$$d_f(2 * 3) = d_f(3) = 2,$$

$$(d_f(2) * 3) \wedge (2 * d_f(3)) = (3 * 3) \wedge (2 * 2) = 0 \wedge 0 = 0,$$

And thus And thus $d_f(2 * 3) \neq (d_f(2) * 3) \wedge (2 * d_f(3))$.

Define an endomorphism f of X by

$f(0) = 0$, $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(4) = 5$, $f(5) = 4$.

Then it is easily checked that d_f is an f-derivation of X.

Remark 5.8. From Example 5.7, we know there is an f-derivation of X which is not a derivation of X. For convenience, we denote $f_x = 0 * (0 * f(x))$ for all $x \in X$. Note that $f_x \in L_p(X)$.

Theorem 5.9. Let d_f be a self-map of a BCIK-algebra X define by $d_f(x) = f_x$ for all $x \in X$. Then d_f is an (l, r)-f-derivation of X. Moreover, if X is commutative, then d_f is an (r, l)-f-derivation of X.

Proof. Let x, y $\in$ X

Since

$$\begin{aligned} 0 * (0 * (f_x * f(y))) &= 0 * (0 * ((0 * (0 * f(x)) * f(y))) \\ &= 0 * ((0 * ((0 * f(y)) * (0 * f(x)))) \\ &= 0 * (0 * (0 * f(y * x))) = 0 * f(y * x) \\ &= 0 * (f(y) * f(x)) = (0 * f(y)) * (0 * f(x)) \\ &= (0 * (0 * f(x))) * f(y) = f_x * f(y), \end{aligned}$$

We have $f_x * f(y) \in L_p(X)$, and thus

$$f_x * f(y) = (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y))),$$

It follows that

$$\begin{aligned} d_f(x * x) = f_{x*x} &= 0 * (0 * f(x*y)) = 0 * (0 * (f(x) * f(y))) \\ &= (0 * (0 * f(x)) * (0 * (0 * f(y))) = f_x * f_y \\ &= (0 * (0 * f_x)) * (0 * (0 * f(y))) = 0 * (0 * (f_x * f(y))) \\ &= f_x * f(y) = (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y))) \\ &= (f_x * f(y)) \wedge (f(x) \wedge f_y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y)), \end{aligned}$$

And so d_f is an (l, r)-f-derivation of X. Now, assume that X is commutative. So $d_f(x) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch x, y $\in$ X, we have

$$\begin{aligned} d_f(x) * f(y) = f_x * f(y) &= (0 * (f_x * f(y))) \\ &= (0 * (0 * f_x)) * (0 * (0 * f(y))) \\ &= f_x * f_x \in V(f_x * f_x), \end{aligned}$$

And so $f_x * f_x = (0 * (0 * f(x))) * (0 * (0 * f_y)) = 0 * (0 * (f(x) * f_y)) = 0 * (0 * (f(x) * d_f(y)) \leq f(x) * d_f(y)$, which implies that $f(x) * d_f(y) \in V(f_x * f_x)$. Hence, $d_f(y) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch, and so

$$\begin{aligned} d_f(x * x) &= (d_f(x) * f(y)) \wedge (f(x) * d_f(y)) \\ &= (f(x) * d_f(y)) \wedge (d_f(x) * f(y)). \end{aligned}$$

This completes the proof.

Proposition 5.10. Let d_f be a self-map of a BCIK-algebra. Then the following hold.

(1) If d_f is an (l, r)-f-derivation of X, then $d_f(x) = d_f(x) \wedge f(x)$ for all $x \in X$.
(2) If d_f is an (r, l)-f-derivation of X, then $d_f(x) = f(x) \wedge d_f(x)$ for all $x \in X$ if and only if $d_f(0) = 0$.

Proof.

(1) Let d_f is an (r, l)-f-derivation of X, Then,

$$d_f(x) = d_f(x * 0) = (d_f(x) * f(0)) \wedge (f(x) * d_f(0))$$
$$= (d_f(x) * 0) \wedge (f(x) * d_f(0)) = d_f(x) \wedge (f(x) * d_f(0))$$
$$= (f(x) * d_f(0)) * ((f(x) * d_f(0)) * d_f(x))$$
$$= (f(x) * d_f(0)) * ((f(x) * d_f(0)) * d_f(0))$$
$$\leq f(x) * (f(x) * d_f(x)) = d_f(x) \wedge f(x).$$

But $d_f(x) \wedge f(x) \leq d_f(x)$ is trivial and so (1) holds.

(2) Let d_f be an (r, l)-f-derivation of X. If $d_f(x) = f(x) * d_f(x)$ for all x Є X, then for x = 0,
$d_f(0) = f(0) * d_f(0) = 0 \wedge f(0) = d_f(0) * (d_f(0) * 0) = 0$.

Conversely, if $d_f(0) = 0$, then $d_f(x) = d_f(x * 0) = (f(x) * (d_f(0)) \wedge (d_f(x) * f(0)) =$
$(f(x) * 0)) \wedge (d_f(x) * 0) = f(x) \wedge d_f(x)$, ending the proof.

Proposition 5.11. Let d_f be an (l, r)-f-derivation of a BCIK-algebra X. Then,

(1) $d_f(x)$ Є $L_p(X)$, then is $d_f(0) = 0 * (0 * d_f(x))$;
(2) $d_f(a) = d_f(0) * (0 * f(a)) = d_f(0) + f(a)$ for all a Є $L_p(X)$;
(3) $d_f(a)$ Є $L_p(X)$ for all a Є $L_p(X)$;
(4) $d_f(a + b) = d_f(a) + d_f(b) - d_f(0)$ for all a, b Є $L_p(X)$.

Proof.

(1) The proof follows from Proposition 5.10(1).
(2) Let a Є $L_p(X)$, then $a = 0 * (0 * a)$, and so $f(a) = 0 * (0 * f(a))$, that is, f(b) Є $L_p(X)$.
Hence

$$d_f(a) = d_f(0 * (0 * a))$$
$$= (d_f(0) * f(0 * a)) \wedge (f(0) * d_f(0 * a))$$
$$= (d_f(0) * f(0 * a)) \wedge (0 * d_f(0 * a))$$
$$= (0 * d_f(0 * a)) * ((0 * d_f(0 * a)) * (d_f(0) * f(0 * a)))$$
$$= (0 * d_f(0 * a)) * ((0 * (d_f(0) * f(0 * a))) * d_f(0 * a))$$
$$= 0 * (0 * (d_f(0) * (0 * f(a))))$$
$$= d_f(0) * (0 * f(a)) = d_f(0) + f(a).$$

(3) The proof follows directly from (2).
(4) Let a, b Є $L_p(X)$. Note that a + b Є $L_p(X)$, so from (2), we note that
$d_f(a + b) = d_f(0) + f(a) + d_f(0) + f(b) - d_f(0) = d_f(a) + d_f(0) - d_f(0)$.

Proposition 5.12. Let d_f be a (r, l)-f-derivation of a BCIK-algebra X. Then,

(1) $d_f(a)$ Є G(X) for all a Є G(X);
(2) $d_f(a)$ Є $L_p(X)$ for all a Є G(X);
(3) $d_f(a) = f(a) * d_f(0) = f(a) + d_f(a)$ for all a, b Є $L_p(X)$;
(4) $d_f(a + b) = d_f(a) + d_f(b) - d_f(0)$ for all a, b Є $L_p(X)$.

Proof.

(1) For any a Є G(X), we have $d_f(a) = d_f(0 * a) = (f(0) * d_f(a)) \wedge (d_f(0) + f(a))$
$= (d_f(0) + f(a)) * ((d_f(0) + f(a)) * (0 * d_f(0))) = 0 * d_f(0)$, and so $d_f(a)$ Є G(X).
(2) For any a Є $L_p(X)$, we get
$d_f(a) = d_f(0 * (0 * a)) = (0 * d_f(0 * a)) \wedge (d_f(0) * f(0 * a))$
$= (d_f(0) * f(0 * a)) * ((d_f(0) * f(0 * a)) * (0 * d_f(0 * a)))$
$= 0 * d_f(0 * a)$ Є $L_p(X)$.

(3) For any a Є $L_p(X)$, we get
$d_f(a) = d_f(a * 0) = (f(a) * d_f(0)) \wedge (d_f(a) * f(0))$
$= d_f(a) * (d_f(a) * (f(a) * d_f(0))) = f(a) * d_f(0)$
$= f(a) * (o * d_f(0)) = f(a) + d_f(a)$.
(4) The proof from (3). This completes the proof.
Using Proposition 5.12, we know there is an (l,r)-f-derivation which is not an (r,l)-f-derivation as shown in the following example.

Example 5.13. Let Z be the set of all integers and "-" the minus operation on Z. Then (Z, -, 0) is a BCIK-algebra. Let $d_f : X \to X$ be defined by $d_f(x) = f(x) – 1$ for all x Є Z.
Then,

$$(d_f(x) – f(y)) \wedge (f(x) - d_f(y)) = (f(x) – 1 – f(y)) \wedge (f(x) – (f(y) – 1))$$
$$= (f(x – Y) – 1) \wedge (f(x – y) + 1)$$
$$= (f(x – Y) + 1) – 2 = f(x – Y) – 1$$
$$= d_f(x - y).$$

Hence, d_f is an (l, r)-f-derivation of X. But $d_f(0) = f(0) – 1 = -1 \neq 1 = f(0) - d_f(0) = 0 - d_f(0)$, that is, $d_f(0) \notin G(X)$. Therefore, d_f is not an (r, l)-f-derivation of X by Proposition 2.12(1).

6. Regular f-derivations

Definition 6.1. An f-derivation d_f of a BCIK-algebra X is said to be a regular if $d_f(0) = 0$
.**Remark 6.2.** we know that the f-derivations d_f in Example 5.5 and 5.7 are regular.
Proposition 6.3. Let X be a commutative BCIK-algebra and let d_f be a regular (r, l)-f-derivation of X. Then the following hold.
(1) Both f(x) and $d_f(x)$ belong to the same branch for all x Є X.
(2) d_f is an (l, r)-f-derivation of X.
Proof.
(1) Let x Є X. Then,

$$0 = d_f(0) = d_f(a_x * x)$$
$$= (f(a_x) * d_f(x)) \wedge (d_f(a_x) * f(x))$$
$$= (d_f(a_x) * f(x)) * ((d_f(a_x) * f(x)) * (f(x) * d_f(a_x)))$$
$$= (d_f(a_x) * f(x)) * ((d_f(a_x) * f(x)) * (f(x) * d_f(a_x)))$$
$$= f_x * d_f(a_x) \text{ since } f_x * d_f(a_x) Є L_p(X),$$

And so $f_x \leq d_f(x)$. This shows that $d_f(x)$ Є V(X), Clearly, f(x) Є V(X).
(2) By (1), we have $f(x) * d_f(y)$ Є $V(f_x * f_y)$ and $d_f(x) * f(y)$ Є $V(f_x * f_y)$. Thus
$d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y)) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$, which implies that d_f is an (l, r)-f-derivation of X.

Remark 6.4. The f-derivations d_f in Examples 5.5 and 5.7 are regular f-derivations but we know that the (l, r)-f-derivation d_f in Example 5.2 is not regular. In the following, we give some properties of regular f-derivations.

Definition 6.5. Let X be a BCIK-algebra. Then define ker d_f = {x Є X / $d_f(x) = 0$ for all f-derivations d_f}.

Proposition 6.6. Let d_f be an f-derivation of a BCIK-algebra X. Then the following hold:
(1) $d_f(x) \leq f(x)$ for all x Є X;

(2) $d_f(x) * f(y) \leq f(x) * d_f(y)$ for all $x, y \in X$;
(3) $d_f(x * y) = d_f(x) * f(y) \leq d_f(x) * d_f(y)$ for all $x, y \in X$;
(4) ker d_f is a sub algebra of X. Especially, if f is monic, then ker $d_f \subseteq X_+$.

Proof.
(1) The proof follows by Proposition 5.10(2).
(2) Since $d_f(x) \leq f(x)$ for all $x \in X$, then $d_f(x) * f(y) \leq f(x) * f(y) \leq f(x) * d_f(y)$.
(3) For any $x, y \in X$, we have

$$d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$$
$$= (d_f(x) * f(y)) * ((d_f(x) * f(y)) * f(x) * d_f(y)))$$
$$= (d_f(x) * f(y)) * 0 = d_f(x) * f(y) \leq d_f(x) * d_f(y),$$

Which proves (3).

(4) Let $x, y \in$ ker d_f, then $d_f(x) = 0 = d_f(y)$, and so $d_f(x * y) \leq d_f(x) * d_f(y) = 0 * 0 = 0$ by (3), and thus $d_f(x * y) = 0$, that is, $x * y \in$ ker d_f, then $0 = d_f(x) \leq f(x)$ by (1), and so $f(x) \in X_+$, that is, $0 * f(x) = 0$, and thus $f(0 * x) = f(x)$, which that $0 * x = x$, and so $x \in X_+$, that is, ker $d_f \subseteq X_+$.

Theorem 6.7. Let be monic of a commutative BCIK-algebra X. Then X is p-semi simple if and only if ker $d_f = \{0\}$ for every regular f-derivation d_f of X.

Proof.
Assume that X is p-semi simple BCIK-algebra and let d_f be a regular f-derivation of X. Then $X_+ = \{0\}$, and So ker $d_f = \{0\}$ by using Proposition 6.6(4), Conversely, let ker $d_f = \{0\}$ for every regular f-derivation d_f of X. Define a self-map d_f of X by $d^*_f(0) = f_x$ for all $x \in X$. Using Theorem 5.9, d^*_f is an f-derivation of X. Clearly, $d^*_f(0) = f_0 = 0 * (0 * f(0)) = 0$, and so d^*_f is a regular f-derivation of X. It follows from the hypothesis that ker $d^*_f = \{0\}$. In addition, $d^*_f(x) = f_x = 0 * (0 * f(x)) = f(0 * (0 * x)) = f(0) = 0$ for all $x \in X_+$, and thus $x \in$ ker d^*_f. Hence, by Proposition 6.6(4), $X_+ \in$ ker $d^*_f = \{0\}$. Therefore, X is p-semi simple.

Definition 6.8. An ideal A of a BCIK-algebra X is said to be an f-ideal if $f(A) \subseteq A$.

Definition 6.9. Let d_f be a self-map of a BCIK-algebra X. An f-ideal A of X is said to be d_f –invariant if
$d_f(a) \subseteq A$.

Theorem 6.10. Let d_f be a regular (r, l)-f-derivation of a BCIK-algebra X, then every f-ideal A of X is $d_f(A) \subseteq A$.

Theorem 6.10. Let d_f be a regular (r, l)-f-derivation of a BCIK-algebra X, then every f-ideal A of X is d_f –invariant.

Proof.
By Proposition 6.10(2), we have $d_f(x) = f(x) \wedge d_f(x) \leq f(x)$ for all $x \in X$. Let $y \in d_f(A)$. Let $y \in d_f(A)$. Then $y = d_f(x)$ for some $x \in A$. It follows that $y * f(x) = d_f(x) * f(x) = 0 \in A$. Since $x \in A$, then $f(x) \in f(A) \subseteq A$ as A is an f-ideal. It follows that $y \in A$ since A is an ideal of X. Hence $d_f(A) \subseteq A$, and thus A is d_f – invariant.

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Theorem 6.11. Let d_f be an f-derivation of a BCIK-algebra X. Then d_f is regular if and only if every f-ideal of X is d_f-invariant.

Proof. Let d_f be a derivation of a BCIK-algebra X and assume that every f-ideal of X is d_f –invariant. Then
Since the zero ideal {0} is f-ideal and d_f –invariant, we have d_f $(\{0\}) \subseteq \{0\}$, which implies that $d_f(0) = 0$.
Thus d_f is regular. Combining this and Theorem 6.10, we complete the proof.

7. Conclusion

Derivation is a very interesting and are of research in the theory of algebraic structures in mathematics. The theory of derivations of algebraic structures is a direct descendant of the development of classical Galosis theory. In this paper, we have considered the notation of f-derivations in BCIK-algebra and investigated the useful properties of the f-derivations in BCIK-algebra. Finally, we investigated the notion of f-derivations in a p-semisimple BCIK-algebra and established some results on f-derivations in a p-semisimple BCIK-algebra. In our opinion, these definitions and main results can be similarly extended to some other algebraic system such as subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras and so forth.

In our future study of f-derivation I BCIK-algebra, may be the following topics should be considered:
(1) To find the generalized f-derivations of BCIK-algebra,
(2) To find more result in f-derivation of BCIK-algebra and its applications,
(3) To find the f-derivations of B-algebras, Q-algebras, subtraction algebras, d-algebra and so forth.

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