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ABSTRACT NAVIER-STOKES EQUATION IN HILBERT SPACE

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Abstract
Our objective here is limited to the Stokes equations alone. The Stokes equations represent the linear portion of the Navier-Stokes equation. We wish to show that, under reasonable conditions, these equations can be reformulated as an abstract linear evolutionary equation $\partial_t u+Au = 0$ on an appropriate Hilbert space. We will show, among other things that the associated Stokes operator A is a positive, self adjoint operator with compact resolvent.

Keywords: Navier-Stokes Equation, Stokes Operator, Helmoltz Operator, Self Adjoint Operator, Hilbert Space.

1. Introduction

The Navier-Stokes equations are deferential equations which, unlike algebraic equations, do not explicitly establish a relation among the variables of interest (e.g. velocity and pressure). Rather, they establish relations among the rates of change. For example, the Navier-Stokes equations for simple case of an ideal fluid (inviscid) can state that acceleration (the rate of change of velocity) is proportional to the gradient (a type of multivariate derivative) of pressure. They are one of the most useful sets of equations because they describe the physics of a large number of phenomena of academic and economic interest. They may be used to model weather, ocean currents, water flow in a pipe, flow around an airfoil (wing), and motion of stars inside a galaxy. As such, these equations in both full and simplified forms are used in the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of the effects of pollution, etc. Coupled with Maxwell's equations they can be used to model and study magneto hydrodynamics.

2. Preliminaries

Let Ω be a bounded open set in R^d , where $d = 2$ or 3, and assume that Ω is of class C^2 . The Stokes equation on Ω are given by

$$\left.\begin{array}{rcl}\partial_t u - \nu\Delta u + \nabla p &=& f,\\ \nabla\cdot u &=& 0\end{array}\right\} \qquad (1)$$

where the velocity $u = (u_1, ..., u_d)$ is a d-dimensional vector field on Ω, the pressure p is a scalar field on Ω, and the forcing function $f = f(t, x)$ is a known or given d-dimensional vector field on Ω. Let $u(t, x) = (u_1(t, x), ..., u_d(t, x))$ be the velocity of the fluid at $(t, x) = (t, x_1, ..., x_d)$, $t \in [0, T]$, $x \in \Omega$, and let $p(t, x)$ denote the pressure at (t, x).

The objective is to solve for $u = u(t, x)$ and $p = p(t, x)$ so that u and p satisfy equation (1) in Ω, and u satisfies the initial value problem

$$u(0, x) = u_0(x),\ x \in \Omega. \qquad (2)$$

Furthermore, we assume that the Dirichlet boundary conditions

$$u(t, x) = 0, \text{ for } x \in \partial\Omega \text{ and } t > 0, \qquad (3)$$

are satisfied, where ∂Ω is the boundary of Ω.

There is the pressure term p and the conservation equation $\cdot u = 0$. Neither of these involve time derivatives. However, by choosing the correct state space H, one address both of these matters: The divergence condition $\cdot u = 0$ is automatically satisfied, and the pressure term p disappear!

We introduce the following function spaces:

$$L^p(\Omega) = L^p(\Omega, \mathbb{R}^d), \qquad \text{for } 1 \le p < \infty \qquad (4)$$

$$H^k(\Omega) = H^k(\Omega, \mathbb{R}^d), \qquad \text{for } k = 1, 2, ..., \infty \qquad (5)$$

$$C^k(\Omega) = C^k(\Omega, \mathbb{R}^d), \qquad \text{for } k = 1, 2, ..., \infty. \qquad (6)$$

Because of the conservation equation $\cdot u = 0$, it is convenient to introduce the following function space.

$$Z_0^\infty \stackrel{def}{=} \{v \in C_0^\infty(\Omega) : \nabla\cdot v = 0 \text{ in } \Omega\}, \qquad (7)$$

The space Z_0^∞ is a collection of divergence-free, smooth vector fields with compact support in Ω. It is a linear subspace of both $L^2 = L^2(\Omega)$ and $H^1(\Omega)$. Next we define

$$H \stackrel{def}{=} Cl_{L^2(\Omega)}(Z_0^\infty), \qquad (8)$$

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$$V \overset{def}{=} Cl_{H^1(\Omega)}(Z_0^\infty) \qquad (9)$$

The inner product $<\cdot,\cdot>_H$ on H is precisely the L^2-inner product $<\cdot,\cdot>_{L^2}$, and the inner product $<\cdot,\cdot>_V$ on V is the H^1-inner product, i.e., from the definition Sobolev space,

$$H^m(\Omega) = W^{m;p}(\Omega) = \{u \in L^p(\Omega) | D^{\sim} u \in L^p(\Omega) \text{ for all } |\alpha| \le m\}. \qquad (10)$$

Let $p = 2$ and $m = 1$, we get

$$H^1(\Omega) = W^{1;2}(\Omega) = \{u \in L^2(\Omega) | D^{\sim} u \in L^2(\Omega) \text{ for all } |\alpha| \le 1\}, u \in H^1(\Omega)$$
$$H^1(\Omega) = \{\int_\Omega |u|^2 dx + \int_\Omega | \quad u|^2 dx\}.$$

It can be written as in terms of inner product

$$< u, v >_V = < u, v >_{L^2} + \Sigma_{i=1}^{d} < D_i u, D_i v >_{L_2}, u, v \in V. \qquad (11)$$

We define $u \; {}^2_{L^2}$ by

$$\|\nabla u\|_{L^2}^2 \overset{def}{=} \sum_{i=1}^{d} \|D_i u\|_{L^2}^2, \quad \text{for} \quad u \in H^1(\Omega). \qquad (12)$$

From the poincare inequality, there is a constant $c > 0$ such that

$$u \;{}_L^2 \le c \quad u \;{}_L^2, \text{ for all } u \quad H_0^1(\Omega), \qquad (13)$$

Substituting the equation (12) in equation (11), we get

$$\|u\|_V^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2. \qquad (14)$$

Substituting the equation (13) in equation (14), we get

$$\begin{aligned} \|u\|_V^2 &\le \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2, \\ \|u\|_V^2 &\le (1+c^2)\|\nabla u\|_{L^2}^2, \\ \|\nabla u\|_{L^2}^2 \le \|u\|_V^2 &\le (1+c^2)\|\nabla u\|_{L^2}^2 \quad \text{for all } u \in V. \end{aligned} \qquad (15)$$

In other words, the norm ‖∇u‖ is equivalent to the V-norm $\|u\|_V$ on V . There is a characterization of H which will be useful. In particular, one has

$$H = Cl_{L^2(\Omega)}\left\{v \in C^1(\bar{\Omega}) : \nabla \cdot v = 0 \text{ in } \Omega \text{ and } v \cdot n = 0 \text{ on } \partial\Omega\right\}. \qquad (16)$$

Indeed, the set

$$Z^1(\Omega) \overset{def}{=} \left\{v \in C^1(\Omega) : \nabla \cdot v = 0 \text{ on } \Omega \text{ and } v \cdot n = 0 \text{ on } \partial\Omega\right\}. \qquad (17)$$

The set $Z_1(\Omega)$ contains Z_0^∞ However, in terms of L^2-norm, the space Z_0^∞ (Ω) is dense $Z_1(\Omega)$. Therefore, the closures of these two spaces in $L^2(\Omega)$ are the same. Recall that if p and u are C^1-functions on Ω, then

$$\begin{aligned} \nabla \cdot (pu) &= u \cdot \nabla p + p\nabla \cdot u, \quad \text{in } \Omega \\ u \cdot \nabla p &= -p\nabla \cdot u + \nabla \cdot (pu) \end{aligned}$$

Since Ω is of class C^2, the Gauss Divergence theorem implies that

$$\int_\Omega u \cdot \nabla p dx = -\int_\Omega p\nabla \cdot u dx + \int_{\partial\Omega} pu \cdot n ds,$$

Where n is the unit outward normal to the boundary ∂Ω. Consequently, we see that if ∇· u=0 in Ω and u · n=0 on ∂Ω, then the above equation becomes

$$\int_\Omega u \cdot \nabla p dx = 0.$$

Let P denote the orthogonal projection of L2(Ω) onto H. P is sometimes referred to as the Helmholtz, or the Leray, Projection. One then has the following result.

Lemma: 2.1. Let Ω be an open, bounded set in R^d of class C2. The $H^{\perp}$, the orthogonal complement of H, satisfies

$$H^\perp = Cl_{L^2(\Omega)}\left\{\nabla p : p \in C^1(\bar{\Omega}, \mathbb{R})\right\}. \qquad (20)$$

Next we want to examine the stationary solutions of the Stokes equations, i.e., the solution of

$$\left.\begin{aligned} -\nu\Delta u + \nabla p &= f, \\ \nabla \cdot u &= 0 \end{aligned}\right\} \qquad (21)$$

With Dirichlet-boundary condition (3). The following result is proved in [2](Theorem3.11).

Lemma:2.2. Let Ω be an open, bounded set in R^d of class C^2. Then there is a constant c > 0 such that, for every $f \in L^2(\Omega) \cap V$ and $p \in H^1(\Omega,R)$ that satisfy (21) with $\int_\Omega p dx = 0,$ and one has

$$\|u\|_{H^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}. \qquad (22)$$

The Stokes operator A is defined by $Au=P(-\Delta u)$ for $u \in D(A)$. Where $D(A)=H^2(\Omega)\cap V$. Thus, if (u,p) is the solution of equation (21) given by the last Lemma, then one has

$$\mathbb{P}(-\nu\Delta u) + \mathbb{P}\nabla p = \mathbb{P}f \quad \text{and} \quad \nu A u = \mathbb{P}f.$$

$$\nabla p = u_n - \mathbb{P}u_n.$$

Applying the projection P on the above equation, we get $\mathbb{P}\nabla p = 0.$
The key, to understanding equation (1) then, is to project this equation into the space H by applying P.

$$\mathbb{P}\partial_t u - \mathbb{P}(\nu\Delta u) + \mathbb{P}\nabla p = \mathbb{P}f,$$

then we obtain

$$\partial_t u + \nu A u = \mathbb{P}f. \qquad (23)$$

Note that Pu = u. Since ∇.u = 0 in Ω and as a result of a Dirichlet boundary conditions, one has u · n=0 on ∂Ω. Therefore $\mathbb{P}\partial_t u = \partial_t u.$

Also note that the pressure term ∇p is missing in equation (23) because P∇p=0, by Lemma 2.1, while the pressure term does not appear in equation (23), it has not been lost. Indeed, if u is given by equation (23), then one uses the equation (20) to find p, see [2]. We now have the following result:

Theorem:2.1. Let Ω be an open, bounded set in R_d of class C^2, and let A be the Stokes operator on Ω. Then the following hold:

(1) The linear operator A is positive and self adjoint.
(2) The inverse A−1 is a compact linear operator on H.
(3) The operator A is a positive, sectorial operator and there exist eigenvalues satisfying ($0 < a = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq$ ......), and the corresponding collection of eigenvalues $\{e_1, e_2, ...\}$ forms an orthogonal basis for H.

Proof. First of all, we show that the Stokes operator A is sysmmetric; i.e., one has

$$< Au, v >_{L^2} = a(u,v) = < u, Av >_{L^2}, \quad \text{for all} \quad u, v \in \mathcal{D}(A). \qquad (24)$$

where *a*(u,v) is the bilinear form

$$a(u,v) = \sum < D_i u, D_i v >_{L^2}, \quad \text{for} \quad u, v \in V. \qquad (25)$$

Indeed, if $u, v \in \mathcal{D}(A)$, then $Pu = u$ and $Pv = v$. Consequently, one has

$$\begin{aligned} < Au, v >_{L^2} &= < -\Delta u, v >_{L^2} \\ &= -\int_\Omega \Delta u \cdot v dx \\ &= -\int_\Omega \nabla^2 u \cdot v dx \end{aligned}$$

By applying Green's formula

$$\begin{aligned} \int_\Omega \nabla^2 u \cdot v dx &= -\int_{\partial\Omega} vn \cdot \nabla u ds + \int_\Omega \nabla u \nabla v dx \\ &= \int_\Omega \nabla u \nabla v dx \\ &= \sum_{i=1}^{d} \int_\Omega D_i u, D_i v dx \end{aligned}$$

$$\begin{aligned} &= a(u,v) \\ &= \int_{\partial\Omega} \nabla vn \cdot u ds - \int_{\Omega} u \cdot \nabla^2 v dx \\ &= -\int_{\Omega} u \cdot \nabla^2 v dx \\ &= -\int_{\Omega} u \cdot \Delta v dx \\ &= < u, -\Delta v >_{L^2} \\ < Au, v >_{L^2} &= < u, Av >_{L^2}. \end{aligned}$$

Hence Stokes operator A is symmetric. Secondly, we show that the bilinear form *a*(u,v) is symmetric and positive. Indeed from the definitions, one has

$$a(u,v) = a(v,u),$$

i.e., *a* is symmetric. From equation (15), then we obtain

$$\begin{aligned} (1+c^2)^{-1}\|u\|_V^2 &\leq \|\nabla u\|_{L^2}^2 \\ &\leq \sum_{i=1}^{d} \|D_i u\|_{L^2}^2 = a(u,v), \quad \text{for all } u \in V. \end{aligned}$$

Hence *a*(u,v) is positive. Thirdly, we show that Stokes operator A is self adjoint and positive. Indeed, if v ∈D(A∗), then there is an f ∈H satisfying Pf =f and

$$< Au, v >_{L^2} = < u, f >_{L^2}, \quad \text{for all } u \in \mathcal{D}(A),$$

since f ∈ H. It follows from Lemma 2.2, that there is a w ∈ D(A) satisfying Aw=f.
We claim that w=v. In order to show this, it suffices to show that

$$< h, w - v >_{L^2} = 0, \quad \text{for all } u \in \mathcal{D}(A).$$

Let h be an arbitrary point in H. We then use Lemma 2.2, to find w ∈D(A) to satisfy Aw =h. One then obtains

$$\begin{aligned} < h, w - v >_{L^2} &= < A\hat{w}, w >_{L^2} - < A\hat{w}, v >_{L^2} \\ &= < \hat{w}, Aw >_{L^2} - < \hat{w}, f >_{L^2} \\ &= < \hat{w}, f - f >_{L^2} \\ &= 0. \end{aligned}$$

It shows that Av = f. Hence D(A∗) ⊂D(A) and A is self adjoint. The positivity of A follows from equation (23).
Finally, we note that the inverseA^{-1} is a compact operator. Indeed, for f ∈H we let u∈D(A) be fixed so that Au=f, i.e., u=A−1f, Then equation (3.2.23) implies that

$$\|A^{-1}f\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}.$$

Since the imbedding $H^2(\Omega) \to L^2(\Omega)$ is compact, we see that A−1 is a compact operator.

3. Navier-Stokes Equations

The Navier-Stokes equations for an incompressible, viscious fluid motion, assume the form

$$\left.\begin{aligned} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f, \\ \nabla \cdot u &= 0 \end{aligned}\right\} \qquad (26)$$

on an open, bounded domain Ω in R_d of class C_2, where d=2 or d=3.

The Stokes operator A arises from the study of linear problem, where the inertial term (u ·∇)u is set equal to zero to obtain the Stokes equations

$$\left.\begin{aligned} \partial_t u - \nu\Delta u + \nabla p = f, \\ \nabla \cdot u = 0. \end{aligned}\right\} \qquad (27)$$

Recall that the Stokes operator arises when one projects equation (26) into the Hilbert space

$$H = \{u \in L^2(\Omega, \mathbb{R}^d) : \nabla\cdot u = 0 \text{ in } \Omega \text{ and } u\cdot n = 0 \text{ on } \partial\Omega\},$$

for Dirichlet boundary conditions. For periodic boundary conditions, one uses

$$H = Cl_{L^2_{per}}(\Omega)\left\{u \in C^\infty_{per}(\Omega) : \nabla\cdot u = 0 \text{ and } \int_\Omega u dx = 0\right\} \qquad (28)$$

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P denote the Helmholtz, or Leray, projection, i.e., the orthogonal projection of the orthogonal complement of H, is

$$H^{\perp} = Cl_{L^2}(\Omega)\left\{\nabla p : p \in C^1(\overline{\Omega}, \mathbb{R})\right\}$$

Thus the pressure satisfies $P\nabla p = 0$ in Ω. We assume that the forcing function $f = f(t)$ satisfies $f \in L^{\infty}(0,\infty;H)$.
In this case one has $Pf = f$. By applying P to equation (25) one obtains

$$\partial_t u + \nu A u = f,$$

where $Au = -\mathbb{P}\Delta u$ and the appropriate boundary condition are satisfied.

$$\|A^{\alpha} u\|^2 \geq \lambda_1^{2\alpha}\|u\|^2, \text{ for } u \in \mathcal{D}(A^{\alpha}) \text{ and } \alpha \in \mathbb{R}.$$

We will also use the interpolation inequality which states that if $\gamma = \theta\alpha + (1-\theta)\beta$, where $\alpha,\beta,\gamma \in R$, $\alpha \geq \beta$, and $0 \leq \theta \leq 1$. Then there is a constant $C>0$ such that

$$\|A^{\gamma} u\| \leq C\|A^{\alpha} u\|^{\theta}\|A^{\beta} u\|^{1-\theta}, \text{ for all } u \in \mathcal{D}(A^{\alpha}).$$

TheHelmholtz projection P can be applied to the Navier-Stokes equations (26), as well,

$$\mathbb{P}\partial_t u - \mathbb{P}(\nu\Delta u) + \mathbb{P}(u \cdot \nabla)u + \mathbb{P}\nabla p = \mathbb{P}f,$$

$$\partial_t u + \nu A u + B(u,v) = f, \quad (29)$$

Which is referred to as the ***Navier-Stokes (evolutionary) equation***, where B(u,v) is the bilinear form

$$B(u,v) \stackrel{def}{=} \mathbb{P}(u \cdot \nabla)v$$

the pressure term p does not appear in equation (29) because $P\nabla p=0$. The pressure can be recovered by applying the complementary projection $(I-P)$ to equation to obtain

$$\nabla p = (I - \mathbb{P})(\nu\Delta u - (u \cdot \nabla)u) + (I - \mathbb{P})f, \quad (30)$$

Where by assumption we have $(I - P)f = 0$. As shown in Lemma 2.1, if one finds a solution $u=u(t)$ of equation (29) on some interval I, then one can use equation (30) to solve for the pressure p. Since the pressure field p is completely determined by the velocity field u, one sometimes refers to p as a slave variable.

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