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THE UNIQUE MINIMAL INVARIANT SET OF SINE'S MAPPING

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Abstract
Minimal invariant sets for sine's mapping share some singular geometrical properties. Here we present some seemingly unknown ones.

Keywords: Sine's Mapping, Fixed Points, Minimal Invariant Sets.

1. Introduction

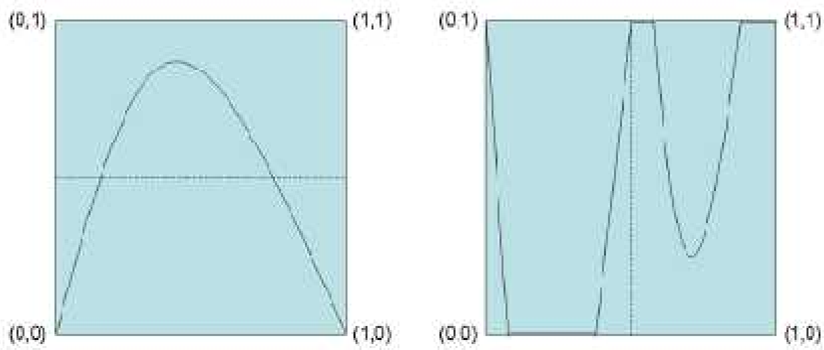
Alspach's mapping is an example of a nonexpansive mapping on a weekly compact, convex subset of $L^1[0,1]$ that is fixed point free. Recall that although we defined Alspach's mapping, T, on

$$C := \{f \in L_1[0,1] : 0 \leq f(x) \leq 1, \forall x \in [0,1]\},$$

it is not fixed point free on this set. It is fixed point free on

$$C_{\frac{1}{2}} := \left\{ f \in C : \|f\|_1 = \frac{1}{2} \right\}.$$

In [4] and [12] modified versions of Alspach's mapping are presented and shown to be fixed point free on all of C. Both of these mappings have a unique minimal invariant set in contrast to Alspach's mapping. In this chapter, we will explore Sine's mapping [12]. Sine showed that the mapping we refer to as "Sine's mapping" (S) is fixed point free on C using techniques similar to those found in [1]. The existence of at least one minimal invariant set is obtain easily using Zorn's lemma. That was essentially the extent of knowledge concerning Sine's mapping prior to this work. Here, we will develop the tools to show that $(1/2)\chi[0,1] \in D_\infty(f)$ for all $f \in C$. This will give us the existence and uniqueness of a minimal S-invariant subset of C without the use of Zorn's lemma. As with Alspach's mapping, there is an iterative method for constructing from below. We will also give some supersets of the minimal invariant set.



The figure to the left represents a function, f, in C and the line y = 1/2. The figure to the right represents Sf and the line x = 1/2.

2. PRELIMINARIES

Recall Alspach's mapping

$$(Tf)(x) := \text{cut}(0,1,2f(2x))\,\chi_{E_{[0,1)}}(x) + \text{cut}(1,2,2f(2x-1))\,\chi_{E_{[1,1)}}(x).$$

Now, we define $S : C \to C$ by

$$S(f) := \chi_{[0,1]} - T(f), \text{ for all } f \in C$$

We will use the properties of the cut function and additional properties here. Let $a,b,c,M \in R$ where $0 \leq a < b \leq M$. Then

$$b - a - \text{cut}(a,b,c) = \text{cut}(M-b, M-a, M-c).$$

The interested reader can verify this property by considering the three cases: $c \leq a$, $c \in (a,b)$, and $c \geq b$. Furthermore, when (1). a, b, c, and M are as above

(2). $f,g \in C$ have disjoint support, and

(3). $\text{supp}(f) \cup \text{supp}(g) = I \subset [0,1]$,

we have the following:

$$
\begin{aligned}
(b-a)\chi_I - \mathrm{cut}(a,b,f+g) &= \mathrm{cut}(M-b, M-a, M\chi_I - f - g) \\
&= \mathrm{cut}(M-b, M-a, M\chi_{supp(f)} + M\chi_{supp(g)} - f - g) \\
&= \mathrm{cut}(M-b, M-a, M\chi_{supp(f)} - f) \\
&\quad + \mathrm{cut}(M-b, M-a, M\chi_{supp(g)} - g) \\
&= \mathrm{cut}(M-b, M-a, M\chi_{[0,1]} - f)\chi_{supp(f)} \\
&\quad + \mathrm{cut}(M-b, M-a, M\chi_{[0,1]} - g)\chi_{supp(g)}.
\end{aligned}
$$

3. ITERATES AND MINIMAL INVARIANT SET

To explore the powers of S, we will first need an auxiliary function. For fixed n $\in$N, take i $\in$N, such that $0 \le i < 2^{2n}$. To define σ, first write

$$i = \sum_{j=0}^{2n-1} d_j 2^j$$

with $d_j \in \{0,1\}$ $\forall$ j, which is a base 2 representation. Then let

$$\sigma_{2n}(i) := \sum_{j=0}^{n-1} d_{2n-2j-2}(i) 2^{2j+1} + \sum_{j=0}^{n-1} \left(1 - d_{2n-2j-1}(i)\right) 2^{2j}.$$

So, $\sigma_2(0) = 1$, $\sigma_2(1) = 3$, $\sigma_2(2) = 0$, and $\sigma_2(3) = 2$..

In order to use induction later, we will need a relationship between σ^{2n} and $\sigma^{2(n+m)}$. Since σ is not defined for odd subscripts, we could remove the 2. However, it is convenient to have the subscript represent the number of digits used in the binary expansions of numbers in the domain. Also, we will see that σ^{2n} is used in the formula for S^{2n}. The needed relationship between σ^{2n}, σ^{2m}, and $\sigma^{2(n+m)}$ is given by the following lemma.

Lemma:3.1. For n,m $\in$N, take j $\in$N, such that $0 \le j < 2^{2m}$ and for k $\in$N with $0 \le k < 2^{2n}$, we have

$$\sigma_{2n+2m}\left(2^{2m}k + j\right) = 2^{2n}\sigma_{2m}(j) + \sigma_{2n}(k).$$

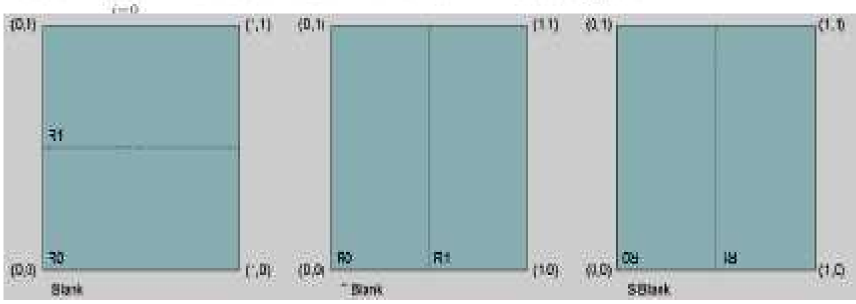
Proof. Before we get too far into the proof, we should note that $d_{2m+l}(2^{2m}k+j) = d_l(k)$, for all $0 \le l < 2n$ and $d_h(2^{2m}k+j) = d_h(j)$, for all $0 \le h < 2m$. Now, consider

$$
\begin{aligned}
&\sigma_{2m+2n}\left(2^{2m}k + j\right) \\
&= \sum_{p=0}^{m+n-1} d_{2(m+n)-2p-2}\left(2^{2m}k + j\right) 2^{2p+1} + \sum_{p=0}^{m+n-1} \left(1 - d_{2(m+n)-2p-1}\left(2^{2m}k + j\right)\right) 2^{2p} \\
&= \sigma_{2n}(k) + \sum_{p=0}^{m-1} d_{2m-2p-2}(j) 2^{2n+2p+1} + \sum_{p=0}^{m-1} \left(1 - d_{2m-2p-1}(j)\right) 2^{2n+2p} \\
&= \sigma_{2n}(k) + 2^{2n}\sigma_{2m}(j)
\end{aligned}
$$

which concludes the proof of Lemma 3.1. Now, we are well prepared to prove the following:

Theorem:3.1. For n$\in$N and f $\in$C,

$$\mathbb{S}^{2n} f(x) = \sum_{i=0}^{2^{2n}-1} cut\left(\sigma_{2n}(i), \sigma_{2n}(i) + 1, 2^{2n} f\left(2^{2n}x - i\right)\right) \chi_{E_{(i,2n)}}(x).$$

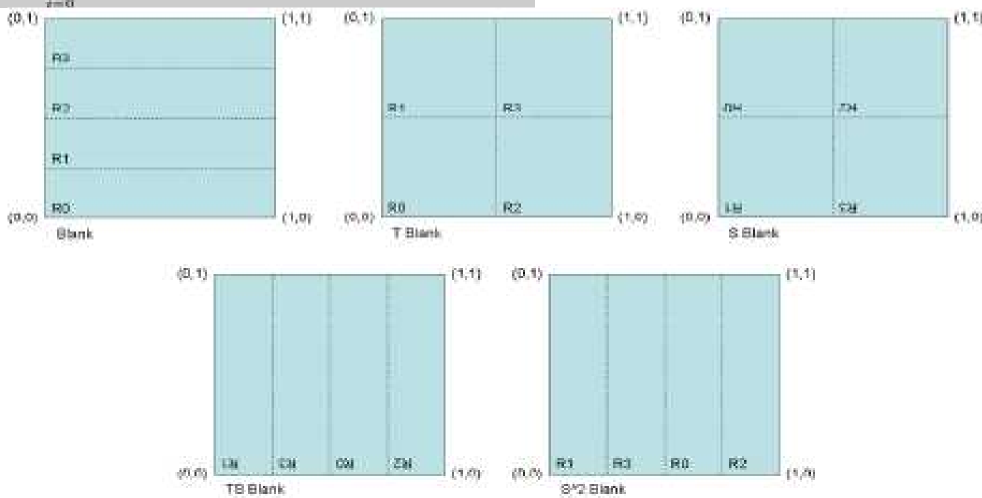


If the graph of a function, f, in C is broken into 2 regions as in the figure to the left, the graph of Tf can be constructed by resizing the portion of the graph of f and translating it to the appropriately labeled position in the figure to the right and adding line segments to ensure that Tf is also in C(center). Flipping the graph about the y= 1/2 axis gives Sf(right). This is denoted by the upside down region labels. Also, S is denoted by S in the figure.

Proof. We will use induction. To begin, we show the theorem holds for n=1. Recall that

$$Tf = \sum_{i=0}^{1} \mathrm{cut}\,(i, i+1, 2f\,(2\mathrm{x}-i))\,\chi_{E_{(i,1)}}.$$

$$\begin{aligned} \mathbb{S}f &= \chi_{[0,1]} - Tf \\ &= \chi_{[0,1]} - \sum_{i=0}^{1} \mathrm{cut}\,(i, i+1, 2f\,(2\mathrm{x}-i))\,\chi_{E_{(i,1)}} \\ &= \sum_{i=0}^{1} \mathrm{cut}\left(1-i, 2-i, 2\chi_{[0,1]} - 2f\,(2\mathrm{x}-i)\right)\chi_{E_{(i,1)}} \end{aligned}$$



$$\begin{aligned} \mathbb{S}^2 f &= \sum_{j=0}^{1} \mathrm{cut}\left(1-j, 2-j, 2\chi_{[0,1]} - 2\mathbb{S}f\,(2\mathrm{x}-j)\right)\chi_{E_{(j,1)}} \\ &= \sum_{j=0}^{1}\sum_{i=0}^{1} \mathrm{cut}\,(2i+1-j, 2i+2-j, 4f\,(4\mathrm{x}-2j-i))\,\chi_{E_{(2j+i,2)}}\chi_{E_{(j,1)}} \\ &= \sum_{k=0}^{3} \mathrm{cut}\,(\sigma_2(k), \sigma_2(k)+1, 4f\,(4\mathrm{x}-k))\,\chi_{E_{(k,2)}}, \end{aligned}$$

$$\mathbb{S}^{2n} f = \sum_{i=0}^{2^{2n}-1} \mathrm{cut}\left(\sigma_{2n}(i), \sigma_{2n}(i)+1, 2^{2n} f\left(2^{2n}\mathrm{x}-i\right)\right)\chi_{E_{(i,2n)}}$$

$$\begin{aligned} \mathbb{S}^{2(n+1)} f &= \mathbb{S}^2\mathbb{S}^{2n} f \\ &= \sum_{j=0}^{3} \mathrm{cut}\left(\sigma_2(j), \sigma_2(j)+1, 4\mathbb{S}^{2n} f\,(4\mathrm{x}-j)\right)\chi_{E_{(j,2)}} \\ &= \sum_{k=0}^{2^{2n+2}-1} cut\left(\sigma_{2n+2}(k), \sigma_{2n+2}(k)+1, 2^{2n+2} f\left(2^{2n+2}\mathrm{x}-k\right)\right)\chi_{E_{(k,2n+2)}}, \end{aligned}$$

where $k=2^{2n}j+i$. This concludes the induction and our proof.

Lemma:3.2. For any $f \in C$ and $s \in S$,

$$\lim_{m\to\infty} \int_{[0,1]} \mathbb{S}^{2m} f \cdot s = \|f\|_1 \int_{[0,1]} s\,.$$

Proof. Since $s \in S$ is a finite sum of constant functions on intervals of the form $E_{(i,n)}$, it suffices to show Lemma 3.2 holds for $s=\chi E_{(l,2n)}$, where $n \in N$ and $0 \le l < 2^{2n}$. Fix $m \in N$. We have that

$$\begin{aligned} &\int_{[0,1]} \mathbb{S}^{2(n+m)} f \chi_{E_{(l,n)}} \\ &= \int_{[0,1]} \left(\sum_{i=0}^{2^{2n+2m}-1} \mathrm{cut}\big(\sigma_{2n+2m}(i), \sigma_{2n+2m}(i)+1, \right. \\ &\qquad \left. 2^{2n+2m} f\left(2^{2n+2m}\mathrm{x}-i\right)\big)\chi_{E_{(i,2n+2m)}} \right)\chi_{E_{(l,2n)}} \end{aligned}$$

From here we wish to reorder the terms in the summation. To that end, define $B := \{j \in N | 0 \le j < 2^{2m}\}$ and

A :={2^{2m}l}+B. Notice that we are summing over A. Now, using Lemma 3.1

$$\sigma_{2m+2n}(A) = \sigma_{2m+2n}(\{2^{2m}l\} + B) = 2^{2n}\sigma_{2m}(B) + \{\sigma_{2n}(l)\}.$$

$$\int_{[0,1]} \mathbb{S}^{2(n+m)} f\chi_{E_{(l,n)}} = \frac{1}{2^{2n+2m}} \sum_{j=0}^{2^{2m}-1} \int_{[0,1]} \operatorname{cut}\left(2^{2n}\sigma_{2m}(j) + \sigma_{2n}(l), 2^{2n}\sigma_{2m}(j) + \sigma_{2n}(l) + 1, 2^{2n+2m} f\right)$$

$$\left|\int_{[0,1]} \mathbb{S}^{2(n+m)} f\chi_{E_{(l_1,2n)}} - \int_{[0,1]} \mathbb{S}^{2(n+m)} f\chi_{E_{(l_2,2n)}}\right| \le \frac{1}{2^{2n+2m}}$$

for $l_1, l_2 \in \mathbb{N}$ with $0 \le l_1 < 2^{2n}$ and $0 \le l_2 < 2^{2n}$. Also, it is easy to verify that

$$\int_{[0,1]} f = \int_{[0,1]} \mathbb{S}^{2n+2m} f = \sum_{k=0}^{2^{2n}-1} \int_{[0,1]} \mathbb{S}^{2n+2m} f\chi_{E_{(k,2n)}}.$$

$$\begin{aligned}
&\left|\int_{[0,1]} \mathbb{S}^{2m+2n} f\chi_{E_{(l,2n)}} - \frac{1}{2^{2n}} \int_{[0,1]} f\right| \\
&= \frac{1}{2^{2n}} \left| 2^{2n} \int_{[0,1]} \mathbb{S}^{2m+2n} f\chi_{E_{(l,2n)}} - \sum_{k=0}^{2^{2n}-1} \int_{[0,1]} \mathbb{S}^{2m+2n} f\chi_{E_{(k,2n)}} \right| \\
&\le \frac{1}{2^{2n}} \sum_{k=0}^{2^{n}-1} \left| \int_{[0,1]} \mathbb{S}^{2m+2n} f\chi_{E_{(l,2n)}} - \int_{[0,1]} \mathbb{S}^{2m+2n} f\chi_{E_{(k,2n)}} \right| \\
&\le \frac{1}{2^{2n}} 2^{2n} \frac{1}{2^{2n+2m}} = \frac{1}{2^{2n+2m}} \to 0, \text{ as } m \to \infty;
\end{aligned}$$

which concludes Lemma 3.2.

Theorem :3.2 *For all $f \in C$, $\mathbb{S}^{2n} f$ converges weakly to $\|f\|_1 \chi_{[0,1]}$ and $\mathbb{S}^{2n+1} f$ converges weakly to $(1 - \|f\|_1)\chi_{[0,1]}$.*
Lemma: 3.3 *For every $f \in C$, $\frac{1}{2}\chi_{[0,1]} \in D_\infty(f)$ and $D_\infty(\frac{1}{2}\chi_{[0,1]}) \subseteq D_\infty(f)$.*

Proof. Take f ∈ C. From Theorem 3.2, S^{2n}f converges weakly to $\|f\|_1 \chi_{[0,1]}$ and $\mathbb{S}^{2n+1} f$ converges weakly to $(1 - \|f\|_1)\chi_{[0,1]}$. So,

$$\|f\|_1 \chi_{[0,1]} \in \overline{conv}\left(\cup_{n\in\mathbb{N}}\{\mathbb{S}^{2n} f\}\right) \subseteq D_\infty(f)$$

$$(1 - \|f\|_1)\chi_{[0,1]} \in \overline{conv}\left(\cup_{n\in\mathbb{N}}\{\mathbb{S}^{2n+1} f\}\right) \subseteq D_\infty(f).$$

$$\frac{1}{2}\chi_{[0,1]} = \frac{1}{2}\|f\|_1 \chi_{[0,1]} + \frac{1}{2}(1 - \|f\|_1)\chi_{[0,1]} \subseteq D_\infty(f).$$

because $D_\infty(f)$ is convex. $D_\infty(f)$ is also $\mathbb{S}$-invariant and closed. Thus, $D_n(\frac{1}{2}\chi_{[0,1]}) \subseteq D_\infty(f)$ for every $n \in \mathbb{N}$ and $D_\infty(\frac{1}{2}\chi_{[0,1]}) \subseteq D_\infty(f)$. □

Theorem: 3.3. $D_\infty((1/2)\chi_{[0,1]})$ is the unique minimal invariant subset of (S,C).
Proof. Obviously, $D_\infty((1/2)\chi_{[0,1]})$ is non-empty, closed, convex, and S-invariant. Suppose M is a non-empty, closed, convex, invariant subset of $D_\infty((1/2)\chi_{[0,1]})$. Choose any f ∈ M. Recall that $D_\infty(f) \subset M$. Lemma 3.3 implies $D_\infty((1/2)\chi_{[0,1]}) \subseteq M$. So, M = $D_\infty((1/2)\chi_{[0,1]})$. Therefore, $D_\infty((1/2)\chi_{[0,1]})$ is a minimal invariant set. Let B⊆C be any minimal invariant set. There is an f ∈B, because B is non-empty. So, $D_\infty((1/2)\chi_{[0,1]}) \subseteq D_\infty(f)$=B. Thus, $D_\infty((1/2)\chi_{[0,1]})$ is the unique minimal invariant set for (S,C).

Theorem: 3.4. Sine’s mapping, S, is fixed point free on C.
Proof. First, recall that the singleton containing any fixed point must be minimal invariant. Now, assume that S has a fixed point in C. Since $D_\infty((1/2)\chi_{[0,1]})$ is the only minimal invariant subset of C by Theorem 3.3, it must be the singleton containing the fixed point. However, $S((1/2)\chi_{[0,1]}) = \chi_{[1/2,1]\ n}$ not equal to $(1/)2\chi_{[0,1]}$, which give the contradiction. Thus, S is fixed point free on C.

4. Discussion of Sine’s mapping

This is possible without using Zorn’s lemma because we have a formula for the iterates of S^2. The formula actually leads to much more that just the removal of a set theoretic axiom. Without a formula for the iterates of S, it is relatively easy to see that all minimal invariant sets of S must be subsets of $C_{1/2}$. However, the number, geometry, and elements of such sets were hard to even guess. Now, the minimal invariant set, $D_\infty((1/2)\chi_{[0,1]})$ can be built from below using the definite of D_∞. Moreover, any invariant superset of $D_\infty((1/2)\chi_{[0,1]})$ can be used to exclude some elements of C from belonging to the minimal invariant set as well. There are similarities between (T,C) and (S,C), and a few important differences. S is actually fixed point free on all of C, whereas T is not. This makes Sine’s mapping somewhat more functionally useful. Also, T has a family of minimal invariant sets, whereas S has a unique minimal invariant set. This makes T a perfect example to have in mind while reading [6], since it explores characteristics of parallel families of minimal invariant sets.

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