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DERIVATIONS ON BP-ALGEBRAS

N. Kandaraj* **A. Arul Devi****
*Associate Professor in Mathematics, Saiva Bhanu Kshatriya College, Aruppukottai
**Assistant Professor in Mathematics, Saiva Bhanu Kshatriya College, Aruppukottai

Abstract
Motivated by some results on derivations on rings, and the generalizations of BCK and BCI algebras, in this paper, we define derivations on BP-algebras and investigate some important results.

Key Words: *BP-Algebras, Derivations on BP- Algebras, Left Derivations on BP-Algebras.*
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1. Introduction

BCK and BCI algebras are two new classes of algebras introduced by Imai and Isaki [3,5].In [1], S.S. Ahn and J.S. Han introduced the notion of BP-algebras. In 2004, Y.B. Jun and X.L. Xin [4] introduced the notion of derivations on BCI algebras. Since then, many authors worked on the notion of derivations on several algebras such as d-algebras [6,7] and TM-algebras [2]. Motivated by this paper we introduce the notion of derivations on BP-algebras.

2. Preliminaries

In this section we recall some basic definitions that are required in our work.

Definition2.1. [3] Let X be a set with a binary operation and a constant 0.Then $(X,*,0)$ is called a BCK -algebras if it satisfies the following axioms:

1. $x * x = 0$
2. $0 * x = 0$
3. $((x * y) * (x * z)) * (z * y) = 0$
4. $(x * (x * y)) * y = 0$
5. $x * y = 0$ and $y * x = 0$ imply $x = y$ $\forall x, y, z \in X$

Definition2.2. [4] Let X be a set with a binary operation* and a constant 0.Then $(X,*,0)$ is called a BCI-algebra if it satisfies the following axioms:

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = 0$ and $y * x = 0$ imply $x = y$ $\forall x, y, z \in X$

Definition2.3. Let x be a BCI-algebra. Two elements x and y in X are said to be comparable if $x \leq y$ or $y \leq x$. Here $x \leq y$ if and only if $x * y = 0$. Also we define $y * (y * x)$ by $x \wedge y$.

Definition2.4. [7] A d -algebra is a non-empty set X with a constant 0 and binary operation $*$ satisfying the following axioms:

1. $x * x = 0$
2. $0 * x = 0$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

Definition2.5. [1] Let X be a set with a binary operation $*$ and a constant 0.

Then $(X,*,0)$ is called a BP-algebra if it satisfies the following axioms.

1. $x * x = 0$
2. $x * (x * y) = y$
3. $(x * z) * (y * z) = x * y$ for any $x, y, z \in X$.

Example 2.6. Let X = {0, 1, 2, 3} ,$(X,*,0)$ be a set with the following cayley table

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X,*,0)$ is a BP - algebra.

Definition2.7: Let X be a d-algebra. A map $\theta: X \to X$ is a left - right derivation (briefly, (l, r) - derivation) on X, if it satisfies the identity

$$\theta(x * y) = (\theta(x) * y) \wedge (x * \theta(y)) \text{ for all } x, y \in X.$$

If θ satisfies the identity

$$\theta(x * y) = (x * \theta(y)) \wedge (\theta(x) * y) \text{ for all } x, y \in X,$$

Then θ is called a right - left derivation (briefly, (r, l) - derivation) on X.

If θ is both an (l,r) and an (r,l) - derivation, then θ is called a derivation on X.

3. Derivations on BP-Algebra

In this section, we define the notion of derivations and composition of derivations on BP - algebras and prove some results. Let $(X,*,0)$ be a BP - algebra. As in the case of BCI - algebras, we use the notion $x \wedge y$ for $y * (y * x)$ $\forall x, y \in (X,*,0)$. We start with the definition of derivation.

Definition3.1: Let X be a BP-algebra. A map $\theta: X \to X$ is a left - right derivation (briefly, (l, r)-derivation) on X, if it satisfies the identity

$$\theta(x * y) = (\theta(x) * y) \wedge (x * \theta(y)) \text{ for all } x, y \in X.$$

If θ satisfies the identity

$$\theta(x * y) = (x * \theta(y)) \wedge (\theta(x) * y) \text{ for all } x, y \in X,$$

Then θ is called a right-left derivation (briefly, (r, l) - derivation) on X.

If θ is both an (l, r) and an (r, l)-derivation, then θ is called a derivation on X.

Example3.2: Consider the BP - algebra given an example. (refer example 2.6)

(1) A self map $\theta: X \to X$ be defined by

$\theta(0) = 3, \theta(1) = 2, \theta(2) = 1, \theta(3) = 0,$ then θ is a derivation on X.

(2) A self map $\theta: X \to X$ be defined by

$\theta(0) = 1, \theta(1) = 0, \theta(2) = 3, \theta(3) = 2,$ then θ is a derivation on X.

(3) A self map $\theta: X \to X$ be defined by

$\theta(0) = 2, \theta(1) = 3, \theta(2) = 0, \theta(3) = 1,$ then θ is a derivation on X.

(4) A self map $\theta: X \to X$ be defined by

$\theta(0) = 0, \theta(1) = 1, \theta(2) = 2, \theta(3) = 3,$ then θ is a derivation on X.

Definition3.3: Let $(X,*,0)$ be a BP-algebra. A self map $\theta: X \to X$ is said to be regular if $\theta(0) = 0.$

Example3.4: Let $(X,*,0)$ be a BP-algebra in example (3.2),(4) is a regular derivaion on X.
One can easily prove the following.

Proposition3.5: Let X be a BP-algebra such that $x * 0 = x \; for all \; x \in X.$ Let θ be a derivation on X.

(1) If θ is a (l,r) - derivation on X, then $\theta(x) = \theta(x) \wedge x, \forall x \in X.$
(2) If θ is a (r,l) - derivation on X, then $\theta(x) = x \wedge \theta(x), \forall x \in X.$

Proposition3.6: Let θ be a self map of a BP - algebra X. Then

$$(x * (x * \theta(x))) * x = (\theta(x) * (\theta(x) * x)) * x$$

Proof:
Since θ is a derivations on X, θ is a (l, r) - derivation on X, Hence the proposition 3.5

$$\theta(x) = \theta(x) \wedge x$$
$$= (x * (x * \theta(x))$$
$$\theta(x) * x = (x * (x * \theta(x)) * x$$

Again $\theta \; is \; a \; (r,l) - derivations$ on X.

$$\Rightarrow \theta(x) = x \wedge \theta(x)$$
$$\theta(x) * x = (\theta(x) * (\theta(x) * (x)) * x$$

Then $(x * (x * \theta(x)) * x = (\theta(x) * (\theta(x) * x)) * x.$

Proposition3.7: Let $(X,*,0)$ be a BP - algebra. Let $\theta: X \to X$ be a derivation.

(1) If $x * \theta(x) = 0, for all \; x \epsilon X,$ then θ is regular.
(2) If $\theta(x) * x = 0, for all \; x \in X$ then θ is regular.

Proposition3.8: Let θ be a self map of a BP - algebra X. If θ is regular (r,l) -derivation on X, then θ is a identity map on X.
Proof:
Given θ is regular (r,l) - derivation on X.

$$\theta(x) = \theta(x * 0)$$
$$=(x * \theta(0)) \wedge (\theta(x) * 0)$$
$$= (\theta(x) * 0) * \left((\theta(x) * 0) * (x * \theta(0))\right)$$
$$= x$$

This shows that θ is the idendity map on X.

Definition3.9: Let X be a BP - algebra and θ_1, θ_2 be two self maps of X. We define

$$\theta_1 \circ \theta_2 : X \to X \ as\ (\theta_1 \circ \theta_2)(x) = \theta_1(\theta_2(x))\ forall\ x \in X.$$

Example 3.10: Let $(X, *, 0)$ be a BP-algebra with the following cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\theta_1(0) = 2, \theta_1(1) = 3, \theta_1(2) = 0, \theta_1(3) = 2,$ is a (l,r) - derivation on X.
$\theta_2(0) = 1, \theta_2(1) = 0, \theta_2(2) = 3, \theta_2(3) = 2,$ is a (l,r) - derivation on X.
$(\theta_1 \circ \theta_2)(0) = 3, (\theta_1 \circ \theta_2)(1) = 2, (\theta_1 \circ \theta_2)(2) = 2, (\theta_1 \circ \theta_2)(3) = 0.$
$(\theta_1 \circ \theta_2)(x)$ is a (l,r) - derivation on X.

Similarly, Composition of two (r, l) - derivation is also a (r, l) - derivation on X The following result gives that the composition of two (l, r) - derivations are again a (r, l) - derivation.

Proposition3.11: Let $(X, *, 0)$ be a BP - algebra. Let $\theta_1 and\ \theta_2$ be two left-right derivations on X.
Then $\theta_1 \circ \theta_2$ is also a left - right derivation on X.
Proof:

$$(\theta_1 \circ \theta_2)(x * y) = \theta_1(\theta_2(x * y))$$
$$= \theta_1\left((\theta_2(x) * y) \wedge (x * \theta_2(y))\right)$$

(using (l, r) - derivations on X)

$$= ((\theta_1\left((\theta_2(x) * y) \wedge (\theta_2(x) * \theta_1(y))\right)$$
$$= \theta_1((\theta_2(x) * y)) \quad (\because x * (x * y) = x \ in\ def\ 2.5)$$
$$= (x * (\theta_1 \circ \theta_2)(y)) * ((x * (\theta_1 \circ \theta_2)(y)) * ((\theta_1 \circ \theta_2)(x) * y))$$
$$= \left((\theta_1 \circ \theta_2)(x) * y\right) \wedge (x * (\theta_1 \circ \theta_2)(y))$$

Hence

$$(\theta_1 \circ \theta_2)(x * y) = \left((\theta_1 \circ \theta_2)(x) * y\right) \wedge (x * (\theta_1 \circ \theta_2)(y))$$

Proposition3.12: Let $(X, *, 0)$ be a BP-algebra. Let $\theta_1 and\ \theta_2$ are (r, l) - derivations on X. Then $\theta_1 \circ \theta_2$ is also a (r, l) - derivation on X.
Proof:

$$(\theta_1 \circ \theta_2)(x * y) = \theta_1(\theta_2(x * y))$$
$$= \theta_1((x * \theta_2(y) \wedge (\theta_2(x) * y)) \text{ (using(r,l)-derivations on X)}$$
$$= \theta_1((x * \theta_2(y))) \qquad (\because x * (x * y) = x \ in\ def\ 2.5)$$
$$= (x * \theta_1(\theta_2(y)) \wedge (\theta_1(x) * \theta_2(y))$$

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$$(\theta_1 \circ \theta_2)(x * y) = (x * \theta_1(\theta_2(y)))$$
$$= (\theta_1(\theta_2(x) * y)) * ((\theta_1(\theta_2(x) * y) * (x * \theta_1(\theta_2(y))$$
$$(\theta_1 \circ \theta_2)(x * y) = (x * (\theta_1 \circ \theta_2)(y)) \wedge ((\theta_1 \circ \theta_2)(x) * y)$$

Hence $(\theta_1 \circ \theta_2)$ *is a* (r,l) − *derivation on* X.

Theorem3.13
Let $(X,*,0)$ *be* a BP-algebra θ_1, θ_2 be two derivations on X, *then* $\theta_1 \circ \theta_2$ is also a derivation on X.

Theorem3.14
Let $(X,*,0)$ *be* a BP-algebra and θ_1, θ_2 are derivations on X. Then $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.

In the following we define the point wise product of two derivations.

Definition3.15
Let X be a BP-algebra and θ_1, θ_2 be two self maps of X, we define
$\theta_1 * \theta_2 : X \to X$ *as* $(\theta_1 * \theta_2)(x) = \theta_1(x) * \theta_2(x), \forall x \in X.$

Theorem3.16
Let X *be* a BP-algebra and θ_1, θ_2 are derivations on X. Then
$\theta_1 * \theta_2 = \theta_2 * \theta_1, \forall x \in X$.
Proof:
Let X *be* a BP-algebra *and* θ_1, θ_2 be two derivations on X.
Since θ_2 is a (l,r) - derivations on X.

$$(\theta_1 \circ \theta_2)(x * y) = \theta_1(\theta_2(x * y))$$
$$= \theta_1\left((\theta_2(x) * y) \wedge (x * \theta_2(y))\right)$$
$$= ((\theta_1\left((\theta_2(x) * y) \wedge (\theta_2(x) * \theta_1(y))\right)$$
$$= \theta_1((\theta_2(x) * y))$$
$$= (\theta_2(x) * \theta_1(y)) \wedge (\theta_1(\theta_2(x)) * y)$$
$$= \left(\theta_1(\theta_2(x) * y) * ((\theta_1(\theta_2(x)) * y) * (\theta_2(x) * \theta_1(y))\right)$$
$$(\theta_1 \circ \theta_2)(x * y) = (\theta_2(x) * \theta_1(y) \dots\dots\dots\dots\dots (1)$$

Also we have (r,l) - derivation on X.

$$(\theta_1 \circ \theta_2)(x * y) = \theta_1(\theta_2(x * y))$$
$$= \theta_1(\theta_2(x) * y) \wedge (x * \theta_2(y))$$
$$= (\theta_1(x * \theta_2(y))$$
$$= (\theta_1(x) * \theta_2(y)) \wedge ((x * \theta_1(\theta_2(y))$$
$$(\theta_1 \circ \theta_2)(x * y) = (\theta_1(x) * \theta_2(y)) \dots\dots\dots\dots\dots (2)$$

From (1) and (2) we get,
$$(\theta_2(x) * \theta_1(y) = (\theta_1(x) * \theta_2(y)) \quad \forall x \in X$$

Putting $y = x$ *we get,*
$$(\theta_2(x) * \theta_1(x) = (\theta_1(x) * \theta_2(x))$$

$$\Rightarrow (\theta_2 * \theta_1)(x) = (\theta_1 * \theta_2)(x)$$
$$\Rightarrow (\theta_2 * \theta_1) = (\theta_1 * \theta_2).$$

4. Left Derivations
In this section we define the notion of the left derivations on BP - algebras, and we prove some results on left derivations on BP-algebras.

Definition4.1
Let X be a BP-algebra. By a left derivation on X, we mean a self map θ of X satisfying

$$\theta(x * y) = (\theta(x) * y) \wedge (\theta(y) * x), \forall x, y \in X$$

Example4.2
Let X={0,1,2} be a BP-algebra with the following cayley table

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

A self map $\theta: X \to X$ be defined by $\theta(0) = 1, \theta(1) = 2, \theta(2) = 0$
Then X is an left derivation on BP-algebra.

Proposition4.3: Let θ be a left derivation on a BP-algebra X. Then for all $x, y \in X$ $we\ have$

(1) $\theta(x) * x = \theta(y) * y$.
(2) $\theta(x * y) = \theta(x) * y$

Proposition4.4: Let θ be a left derivation on a BP-algebra X. Then θ is regular, if and only if

$$\theta(x) \leq x, \forall x \in X.$$

Proof:
Now,
$$\theta(0) = \theta(x * x)$$
$$= (\theta(x) * x) \wedge (\theta(x) * x)$$
$$= \theta(x) * x$$

Let θ be regular. Since $\theta(0) = 0, \theta(x) * x = 0\ for all\ x \in X.$
Which implies $\theta(x) \leq x,\ \forall x \in X.$
Conversely, assume that $\theta(x) \leq x,\ \forall\, x \in X.$

$$\Rightarrow \theta(x) * x = 0 \Rightarrow \theta(0) = 0$$

Hence θ is regular.

Proposition4.5: Let X be a BP-algebra and θ be a left derivation on X, then

$$\theta(x) = x \wedge \theta(x)\ and\ \theta(x) = \theta(x) \wedge x.$$

Proof:
Let X be a BP-algebra.

$$x \wedge \theta(x) = \theta(x) * (\theta(x) * x)$$

$$= \theta(x) * 0 \ (\because \theta(x) \leq x \ , \forall x \in X \Rightarrow \theta(x) * x = 0)$$
$$= \theta(x)$$

Also, $\theta(x) \wedge x = x * (x * \theta(x)) = \theta(x)$

Definition4.6

Let X be a BP-algebra and θ_1, θ_2 be two self maps of X. We have

$$\theta_1 \circ \theta_2 : X \to X \ as \ (\theta_1 \circ \theta_2)(x) = \theta_1(\theta_2(x)), \forall x \in X.$$

The following theorem shows that the composition of two left - derivations is again a left derivation on X.

Theorem4.7

Let $(X,*,0)$ be a BP-algebra. Let θ_1, θ_2 be two derivations on X, then $\theta_1 \circ \theta_2$ is also a left derivation on X.

Proof:

Given θ_1 is a left derivation on X.

$$\theta_1(x * y) = (\theta_1(x) * y) \wedge (\theta_1(y) * x)$$

Similarly, θ_2 is a left derivation on X.

$$\theta_2(x * y) = (\theta_2(x) * y) \wedge (\theta_2(y) * x)$$

Now,

$$\begin{aligned}(\theta_1 \circ \theta_2)(x * y) &= \theta_1(\theta_2(x * y)) \\ &= \theta_1((\theta_2(x) * y)) \\ &= (\theta_1(\theta_2(x)) * y)) \\ &= (\theta_1(\theta_2(x) * y)) \wedge (\theta_1(\theta_2(y) * x)) \\ &= (\theta_1 \circ \theta_2)(x) * y \wedge (\theta_1 \circ \theta_2)(y) * x)\end{aligned}$$

Hence $\theta_1 \circ \theta_2$ is a left derivation on X.

We observe that the composition of regular left derivations are commutative as seen below.

Theorem4.8

Let $(X,*,0)$ $be\ a\ BP - algebra$ and θ_1, θ_2 are regular left derivations on X.

Then $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.

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