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COUNTING NUMBER OF VISIBLE POINTS IN A RECTANGLE USING MOBIOUS μ-FUNCTION

Dr.A.Vethamanickam* Mrs.K.E.Usha Nirmala Kumari**
**Associate Professor,Rani Anna Govt.College for women,Thirunelveli.*
***Research Scholar,H.H.The Rajah's College,Pudukkottai.*

Abstract
We say that one lattice point is visible from another if no third lattice point lies on the line joining them. A lattice point visible from the origin is called a visible point. To find the visible points is very useful in probability theory. In this paper we deal with counting visible points in rectangular region using Mobious μ- function.

Introduction

The average order of the mobious function : We are interested in the behaviour of $\mu_a(n) = \frac{1}{n}\sum_{k=1}^{n}\mu(k)$This is a horse of a completely different colour, as we are summing up the values 0 and $\pm$1.We just saw that μ is nonzero a positive proportion, namely $6/\pi^2$ of the time.Looking at values of the Mobious function on square free integers one finds that it is indeed $\pm$1about as often as it is -1.

Which means that there ought to be a lot of cancellation in the sum. If every single term in the sum were 1 then $\mu_a(n)$ would still only be equal to 1, and similarly if every single term were -1 the average order would be -1, so the answer (if the limit exists 1) is clearly somewhere in between. Let's think a little more. Restricting do square free numbers (as a $\frac{6}{\pi^2}$ proportion are), it is not hard to believe that relatively few integers are divisible only by a fixed number of primes (we will prove a few results in this direction) (after on) in other words, for large N, most square integers $1 \leq n \leq N$ will have lots of prime factors and guessing whether they have an even or odd number of factors seems like guessing whether a large number is even or odd without any further information, the most obvious guess is that $\mu(n) = +1$ about as often as it equals -1.

Theorem: 1.1

We have $\lim_{N \to \infty} \frac{V(N)}{L(N)} = 6/\pi^2$

Proof

We are asking after all for the number of ordered pairs of integers (x, y) each of absolute value of most N, with x and y relatively prime.We can make the probability as close to $6/\pi^2$ as we wish by taking N sufficiently large.We observe that the eight lattice points immediate nearest the origin. (ie) those with max (|x|, |y| $\leq$1. are all visible.The total number of visible lattice points in the square |x|, |y| < N will then be these 8 plus 8 times. The number of lattice points with $2 \leq x \leq N,\ 1 \leq y \leq x$. (ie) the ones whose angular coordinate θ satisfy 0<$\theta \leq \frac{\pi}{2}$.

But now we have

$$V(N): 8 + \sum_{2 \leq \eta \leq N} \sum_{1 \leq m \leq n(m,n)} 1 = 8\Sigma(\varphi(n))_{1 \leq n \leq N}$$

But Now L(N) = (2N+1) $\sim$ 4N^2 and

$$8\sum_{n=1}^{N}{}' \varphi(n) \sim \frac{3}{\pi^2} N^2, so\ that$$

$$\frac{V(N)}{L(N)} = \frac{8}{(2N+1)^2} + \frac{8\sum_{n=1}^{N} \varphi(n)}{(2n+1)^2}$$

$$= 0 + 8.\frac{\left(\frac{3}{\pi^2}\right)}{4}$$

$$= 6/\pi^2$$

Theorem:1. 2

The number N' (X, Y) of visible points in Q(x, y) is exactly.

$\sum_{k=1}^{\infty} \mu(k) \left[\frac{x}{k}\right]\left[\frac{y}{k}\right]$. Here the sum is finite, since the terms are zero when k>min(x,y).

Proof:

We know that:

The average order of the mobious function is zero

$$\lim_{x \to \infty.} \frac{\sum_{k=1}^{n} \mu(k)}{n} = 0$$

Which is equivalent to the statement that the sum $\sum_{k=1}^{n} \mu(k)$ is of smaller order than ‘n’ itself.
But if we to some computations you will see that these partial sums seem in practice to be quite a bit smaller than n, and also we know that, μ is non zero a positive proportion, namely $6/\pi^2$ of the time, The mobius property we use is that.

$$\sum_{d/n} \mu(d) = \left[\frac{1}{n}\right]$$

We have the N' (x, y) visible points in Q(X, Y) is.

$$N'(X, Y) = \sum_{\substack{m \le X \\ n \le y \\ (m,n)=1}} 1$$

$$= \sum_{\substack{m \le X \\ n \le y}}{}' \sum_{\substack{d \\ (m,n)}}{}' \mu(d)$$

$$= \sum_{d=1}^{\infty} \mu(d). \sum_{\substack{\frac{d}{m}, m \le X \\ \frac{d}{n}, n \le y}} 1$$

Let m = kd, n = ld, then

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The limits are

(i) $m \leq x \Longrightarrow m = x$

$$k = \frac{x}{d}$$

(ii) $n \leq y\ n = y$

$1 = \frac{y}{d}$

N'(x,y)= $\sum_{d=1}^{\infty} \mu(d) \sum_{k\leq \frac{x}{d}} 1 \ \sum_{l\leq \frac{y}{d}} 1$

$$= \sum_{d=1}^{\infty} \mu(d) \lfloor e \rfloor^{x/d} \lfloor f \rfloor^{y/d}$$

$N^1(x,y) = \sum_{d=1}^{\infty} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor \left\lfloor \frac{y}{d} \right\rfloor.$

From the above theorem (1.1)

The fraction of visible point is $\frac{N'(x,y)}{\lfloor x \rfloor \lfloor y \rfloor}$,

we can deduce the "rectangle limit" which is equivalently to

$$\lim_{x, y \to \infty} \frac{1}{xy} \ N'(X, Y) = \frac{6}{\pi^2}$$

As x and y increase independently.

Theorem:1.3

$$\frac{1}{xy} \ N'(X, Y) = \frac{6}{\pi^2} + 0 \left(\frac{\log z}{z}\right)$$

Where z = min (x ,y)

Proof:

If $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$,

$\Longrightarrow \lfloor \alpha \rfloor = \alpha - \lfloor \alpha \rfloor$ → (1)

Let $\alpha = \frac{x}{y} . \frac{y}{k}$

From Equation (1)

$N_{(x,y)} = \left\lfloor \frac{x}{k} \right\rfloor \left\lfloor \frac{y}{k} \right\rfloor = \frac{x}{k} . \frac{y}{k} - \left\{\frac{x}{k}\right\} \frac{y}{k} - \left\{\frac{y}{k}\right\} \frac{x}{k} + \left\{\frac{x}{k}\right\} \left\{\frac{y}{k}\right\}$ → 2

We know that the above theorem (3.5)

The no of $N^1(x, y)$ of visible points in Q(x, y) is exactly.

$\sum_{k=1}^{\infty} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor \left\lfloor \frac{y}{k} \right\rfloor$, where k > min (x,y)

multiple by $\mu(k)$ and sum on k from 1to z on both sides

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$$\frac{N'(x,y)}{xy} = \sum_{k\le z} \frac{\mu(k)}{xy}\frac{xy}{k^2} - \frac{1}{xy}\sum_{k\le z} \mu(k)\left\{\frac{x}{k}\right\}\frac{y}{k}$$

$$-\frac{1}{xy}\sum_{k\le z}\mu(k)\left\{\frac{y}{k}\right\}\frac{x}{k} + \frac{1}{xy}\sum_{k\le z}\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\}$$

$$\frac{N'(x,y)}{xy} = \sum_{k\le z}\frac{\mu(k)}{k^2} - \frac{1}{x}\sum_{k\le z}\frac{\mu(k)}{k}\left\{\frac{x}{k}\right\}$$

$$-\frac{1}{y}\sum_{k\le z}\frac{\mu(k)}{k}\left\{\frac{y}{k}\right\} + \frac{1}{xy}\sum_{k\le z}\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\} \qquad \rightarrow 3$$

From the above theorem and the first term on the right tends to

$$\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2} = \frac{6}{\pi^2} \text{ as } \quad z \rightarrow \infty \qquad \rightarrow A$$

Then the remaining term is

(ie) the tail of the series is 0(log z/z)

The absolute value of the sum of the three remaining terms is less than

Therefore

$$\frac{1}{x}\sum_{k\le z}\frac{\mu(k)}{k}\left\{\frac{x}{k}\right\} + \frac{1}{y}\sum_{k\le z}\frac{\mu(k)}{k}\left\{\frac{y}{k}\right\} + \frac{1}{xy}\sum_{k\le z}\mu(k)\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\}$$

$$\left(\frac{1}{x}+\frac{1}{y}\right)\left[\sum_{k\le z}\left[\frac{\mu(k)}{k}\right]\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\} + \frac{1}{xy}\sum_{k\le z}\mu(k)\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\}\right]$$

$$\left(\frac{1}{x}+\frac{1}{y}\right)\left[\sum_{k\le z}\frac{1}{k} + \frac{1}{xy}\sum_{k\le z}1\right], \qquad \mu(k)\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\} \text{ is minimum.}$$

The sum of the limit value $k \le z$, so

$$\le \frac{1}{x}+\frac{1}{y}\left[\sum_{k\le z}\frac{1}{k} + \frac{1}{xy}\sum_{k\le z}1\right]$$

$$\le \frac{x+y}{xy}\left[\sum_{k\le z}\frac{1}{k} + \frac{1}{xy}\sum_{k\le z}1\right]$$

$$\le \frac{x+y}{xy}\sum_{k\le z}\frac{1}{k} + \frac{x+y}{xy}\cdot\frac{1}{xy}\sum_{k\le z}1 \qquad \rightarrow 4$$

Put xy = z and x, y = 1 in (4)

$$\le \frac{2}{z}\sum_{k\le z}\frac{1}{k} + \frac{2}{z}\cdot\frac{1}{z}\sum_{k\le z}1$$

$$\leq \frac{2}{z}\sum_{k\leq z}\frac{1}{k}+\frac{2}{z^2}\sum_{k\leq z}1$$

The integrating value of 1/k and the order.

$$= \frac{2}{z}O(\log z)+\frac{2}{z^2}.z$$

$$= \frac{2}{z}O(\log z)+\frac{2}{z}$$

$$= \frac{2}{z}O(\log z+1)$$

$= O\left(\frac{\log z}{z}\right)$Combined the equation A & B

$$\frac{N'(x,y)}{xy}=\frac{6}{\pi^2}+O\left(\frac{\log z}{z}\right)$$

This value is obtained by counting the fraction of visible points in an expanding region of much more general shape. (ie) rectangle.

Theorem 1.4

The fraction of points (m, n) in Q(x, y) such that (m, n) = k tends of 6/$k^2.1/k^2$,as x, y $\rightarrow \infty$

Proof:

Consider if a lattice point is selected at random in two dimensions, the probability that it is visible from the origin is $\frac{6}{\pi^2}$.This is also the probability that two integers picked at random are relatively prime.

We use this results in the theorem, as follows, that, the condition.

(m, n) = k holds iff m = km', n =kn'

Where (m', n') = 1. Such that The number of points in Q(x, y) with (m, n) = k in equal to the number N'(x/k, y/k) of visible points in $\left[Q\left(\frac{x}{k},\frac{y}{k}\right)\right]$

Using the above them equation (3)

$$\frac{N'(x,y)}{xy}=\sum_{k\leq z}{}'\frac{\mu(k)}{k^2}-\frac{1}{x}\sum_{k\leq z}{}'\mu(k)\left\{\frac{x}{k}\right\}$$
$$-\frac{1}{y}\sum_{k\leq z}{}'\mu(k)\left\{\frac{y}{k}\right\}+\frac{1}{xy}\sum_{k\leq z}{}'\mu(k)\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\}$$

$$N'(x,y)=\sum_{k=1}^{\infty}{}'\mu(k)\left\{\frac{x}{k}\right\}\left\{\frac{y}{k}\right\} \text{ and}$$

$$\frac{1}{xy}N'\left(\frac{x}{k},\frac{y}{k}\right)=\frac{1}{k^2}\cdot\frac{1}{\frac{x}{k}\cdot\frac{y}{k}}N'\left(\frac{x}{k}\right)\left(\frac{y}{k}\right)$$

$$=\frac{1}{k^2}\cdot\frac{6}{\pi^2}\text{as } x,y\rightarrow\infty$$

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$$\frac{1}{xy} N'\left(\frac{x}{k}, \frac{y}{k}\right) = \frac{1}{k^2} \cdot \frac{6}{\pi^2}$$

This results could be described by saying that the probability that two randomly selected integers m and n have g.c.d equal to k.

(ie) $$k = \frac{6}{\pi^2} \cdot \frac{1}{k^2}$$

Result: In this way we can find the visible points in a rectangles using the Mobious μ- function.This result has been obtained in the case of the square if

x = y by based on the asymptotic formula $\lim_{n \to \infty} \frac{\mu(n)}{f(n)} = 1 \quad \mu n \sim f(n)$.

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