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SOME RESULTS ON UNION OF TWO FUZZY GRAPHS

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Abstract
In this paper, we find the order, size and degree of a vertex in the union of two fuzzy graphs. Strong, complete, regular and connected nature of union of two fuzzy graphs have been studied.

Keywords: Fuzzy graphs, union of two fuzzy graphs, strong fuzzy graphs, complete fuzzy graph, regular fuzzy graph, connected complete and regular fuzzy graphs.

1. Introduction

Fuzzy graph was introduced say Rosen field in 1975. Fuzzy graphs can be used in traffic light problems, time table scheduling etc. The operators union, join, cartesian product and composition of two fuzzy graphs were defined by Mordeson and Peng [9]. Later, Nirmala and Vijaya [11] determined the degree of a vertex in new fuzzy graphs obtained from two fuzzy graphs using the operation cartesian, tensor, normal product and composition on two fuzzy graphs we find the order, size and degree of a vertex in the union of two fuzzy graphs. We have illustrated an example that the union of two strong fuzzy graphs need not be a strong fuzzy graph. We have proved that the necessary and the sufficient condition for the union of two complete fuzzy graphs to be a complete fuzzy graph.

2. Preliminaries

A fuzzy graph G is a pair of function (σ, μ) where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ. The underlying crisp graph of G: (σ,μ) is denoted by $G^*(V, E)$ where $E \subseteq V \times V$, G: (σ, μ) is called connected fuzzy graph if for all u, v $\in$ V there exists at least one non zero path between u and v. G is called strong fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $(u, v) \in E$ and complete fuzzy graph of $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, $\forall$ u, v $\in$ V. The degree of a vertex is of G: (σ,μ) is defined as $d_G(u) = \sum_{uv\in E} \mu(u\ v)$. The order of a fuzzy graph G:(σ,μ) is defined as $O(G) = \sum_{u\in V} \pm(u)$. The size of fuzzy graph G (σ,μ) is defined as $q(G) = \sum_{uv\in E} \mu(u,v)$.If $d_G(v) = k$, for all $v\in G$. G is said to be a regular fuzzy graphs of degree 'k' or k-regular

3. Union of Two Fuzzy Graphs

Definition 3.1. Let G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* :(V_2, E_2)$ respectively. Let $V=V_1\cup V_2$ and let $E=\{uv/u,v\in V, uv\in E \text{ or } uv\in E_2 \text{ or } uv\in E_1\cap E_2\}$then the **union** of G_1 and G_2 denoted by $G_1\cup G_2$: $(\sigma_1\cup\sigma_2, \mu_1\cup\mu_2)$ is defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u), & \text{if} \quad u \in V_1 - V_2 \\ \sigma_2(u), & \text{if} \quad u \in V_2 - V_1 \\ \sigma_1(u) \vee \sigma_2(u), & \text{if} \quad u \in V_1 \cap V_2 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u,v) = \begin{cases} \mu_1(u,v), & \text{if} \quad uv \in E_1 - E_2 \\ \mu_2(u,v), & \text{if} \quad uv \in E_2 - E_1 \\ \mu_1(u,v) \vee \mu_2(u,v), & \text{if} \quad uv \in E_1 \cap E_2 \end{cases}$$

Theorem 3.2. Let G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) be two fuzzy graphs with underlying crisp graphs $G_1^* :(V_1, E_1)$ and $G_2^* :(V_2, E_2)$.

Then $(G_1 \cup G_2)$: $((\sigma_1 \cup \sigma_2), (\mu_1 \cup \mu_2))$ is a fuzzy graph.

Proof:

We shall prove the theorem in three cases.

Case 1: If $uv \in E_1 - E_2$

$$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u\, v)$$
$$\leq \sigma_1(u) \wedge \sigma_1(v)$$
$$\leq [\sigma_1(u) \vee \sigma_2(u)] \wedge [\sigma_1(v) \vee \sigma_2(v)]$$
$$= (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$$
$$\therefore \quad (\mu_1 \cup \mu_2)(u, v) \leq (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$$

Case 2: If $uv \in E_2 - E_1$, we can prove as in case (1)

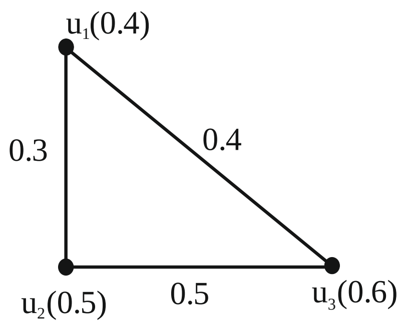
Case 3: If $uv \in E_1 \cap E_2$

$$(\mu_1 \cup \mu_2)(u, v) = \mu_1(uv) \vee \mu_2(uv)$$
$$\leq [\sigma_1(u) \wedge \sigma_1(v)] \vee [\sigma_2(u) \wedge \sigma_2(v)]$$
$$\leq [\sigma_1(u) \vee \sigma_2(u)] \wedge [\sigma_1(v) \vee \sigma_2(v)]$$
$$= (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$$
$$\therefore (\mu_1 \cup \mu_2)(u, v) \leq (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$$

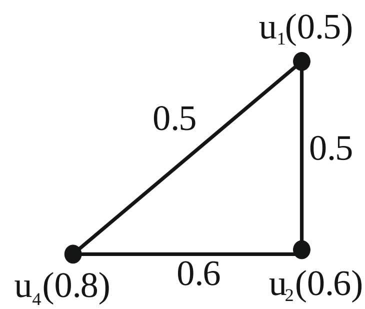
$\therefore$ From above three cases we conclude that the union of two fuzzy graphs is again a fuzzy graph.

The following example illustrate that the union of two fuzzy graphs.

Example: 3.3. ■



G_1: (σ_1, μ_1) fuzzy graph



G_2: (σ_2, μ_2) fuzzy graph

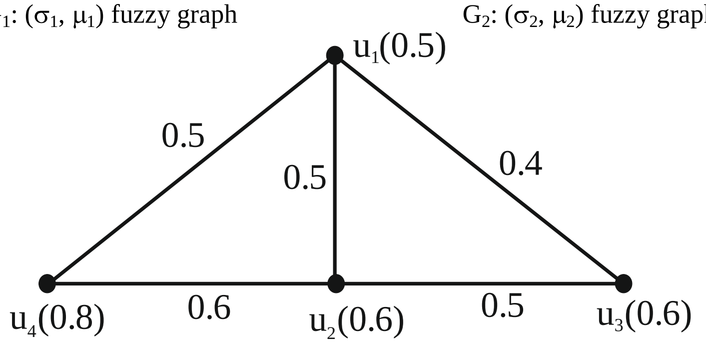


Figure 1 : $G_1 \cup G_2$: $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ fuzzy graph

4. Order of Union of Two Fuzzy Graphs

Definition: 4.1. Let $G_1 \cup G_2$: $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ be the union of two fuzzy graphs $G_1(\sigma_1, \mu_1)$ and G_2: (σ_2, μ_2) then the **order** of $G_1 \cup G_2$ denoted by $O(G_1 \cup G_2)$ is given by

$$O(G_1 \cup G_2) = \sum_{u \in V_1 - V_2} \sigma_1(u) + \sum_{u \in V_2 - V_1} \sigma_2(u) + \sum_{u \in V_1 \cap V_2} \sigma_1(u) \vee \sigma_2(u)$$

The following theorem gives the relation between the order of G_1, order of G_2 and order of $G_1 \cup G_2$

Theorem 4.2. Let G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) be two fuzzy graphs. Then,

$$O(G_1 \cup G_2) = O(G_1) + O(G_2) - \sum_{u \in V_1 \cap V_2} [\sigma_1(u) \wedge \sigma_2(v)]$$

Proof:

Let $V_1 = \{u_1, u_2, \ldots\ldots. u_m, \; x_1, x_2, \ldots\ldots\ldots. x_p\}$

$V_2 = \{v_1, v_2, \ldots\ldots. v_n, \; x_1, x_2, \ldots\ldots\ldots. x_p\}$

Then the order of G_1 is

$$O(G_1) = \sum_{i=1}^{m} \sigma_1(u_i) + \sum_{k=1}^{p} \sigma_1(x_k)$$

and the order of G_2 is

$$O(G_2) = \sum_{j=1}^{m} \sigma_2(v_j) + \sum_{k=1}^{p} \sigma_2(x_k)$$

By the definition of union of two fuzzy graphs

If $x_k \in V_1 \cap V_2$, then

$(\sigma_1 \cup \sigma_2)(x_k) = \sigma_1(x_k) \vee \sigma_2(x_k)$, $k = 1, 2. . . p$

If $u_i \in V_1 - V_2$, then

$(\sigma_1 \cup \sigma_2)(u_i) = \sigma_1(u_i)$, $i = 1, 2. . . m$

If $v_j \in V_2 - V_1$, then

$(\sigma_1 \cup \sigma_2)(u_j) = \sigma_2(u_j)$, $j = 1, 2. . . n$

Now $O(G_1) + O(G_2)$ $= \sum_{i=1}^{m} \sigma_1(u_i) + \sum_{j=1}^{n} \sigma_2(v_j) + \sum_{k=1}^{p} \sigma_1(x_k) + \sum_{k=1}^{p} \sigma_2(x_k)$

$$= \sum_{i=1}^{m} \sigma_1(u_i) + \sum_{j=1}^{n} \sigma_2(u_j) + \sum_{k=-1}^{p} \sigma_1(x_k) \vee \sigma_2(x_k)$$

$$+ \sum_{k=1}^{p} \sigma_1(x_k) \wedge \sigma_2(x_k)$$

$$= O(G_1 \cup G_2) + \sum_{k=1}^{p} \sigma_1(x_k) \wedge \sigma_2(x_k)$$

$$O(G_1 \cup G_2) = O(G_1) + O(G_2) - \sum_{k=1}^{p} \sigma_1(x_k) \wedge \sigma_2(x_k)$$

Hence the theorem.

The result $O(G_1 \cup G_2) = O(G_1) + O(G_2) - \sum_{k=1}^{p} \sigma_1(x_k) \wedge \sigma_2(x_k)$ can be verified for the following example.

5. Size of Union of Two Fuzzy Graphs

Definition 5.1. Let $G_1 \cup G_2$: $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ be the union of two fuzzy graphs G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2). Then its **size** denoted by $q(G_1 \cup G_2)$ is given by

$$q(G_1 \cup G_2) = \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_2 - E_1} \mu_2(uv) + \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \vee \mu_2(uv)$$

The following theorem gives the relation between the sizes of G_1 and G_2 and the size of $G_1 \cup G_2$.

Theorem 5.2. Let G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) be two fuzzy graphs. Then

$$q(G_1 \cup G_2) = q(G_1) + q(G_2) - \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv)$$

proof: similarly as above.

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6. Degree of A Vertex in The Union of Two Fuzzy Graphs

The degree of any vertex in the union of two fuzzy graphs can be found by using following theorem

Theorem 6.1. For any vertex $v \in V_1 \cup V_2$. Then

$$d_{G_1 \cup G_2}(u) = \begin{cases} d_{G_1}(u), \text{ if } u \in V_1 - V_2 \\ d_{G_2}(u), \text{ if } u \in V_2 - V_1 \\ d_{G_1}(u) + d_{G_2}(u), \text{ if } u \in V_1 \cap V_2 \text{ and the edges incident at u does not lie in } E_1 \cap E_2 \\ d_{G_1}(u) + d_{G_2}(u) - \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv), \text{ if } u \in V_1 \cup V_2 \text{ and the edges} \\ \qquad \text{incident at u lies in } E_1 \cap E_2 \end{cases}$$

Proof:

Any vertex in the union of two fuzzy graphs can be in any one of the following cases.

Case (1): If $u \in V_1 - V_2$ and the edges $(u, v) \in E_1 - E_2$

$$d_{G_1 \cup G_2}(u) = \sum_{uv \in E_1 - E_2} \mu_1(uv) = d_{G_1}(u)$$

Case (2): If $u \in V_2 - V_1$ and the edges $(u, v) \in E_2 - E_1$

$$d_{G_1 \cup G_2}(u) = \sum_{uv \in E_2 - E_1} \mu_2(uv) = d_{G_2}(u)$$

Case (3): If $u \in V_1 \cap V_2$ and the edges (u, v) lies either in E_1 or in E_2 but not in both.

$$d_{G_1 \cup G_2}(u) = \sum_{uv \in E_1} \mu_1(uv) + \sum_{uv \in E_2} \mu_2(uv)$$

$$= d_{G_1}(u) + d_{G_2}(u)$$

Case 4: If $u \in V_1 \cap V_2$ and the edges $uv \in E_1 \cup E_2$ (some edges belongs to $E_1 \cap E_2$)

$$d_{G_1 \cup G_2}(u) = \sum_{uv \in E} (\mu_1 \cup \mu_2)(uv)$$

$$= \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_2 - E_1} \mu_2(uv) + \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \vee \mu_2(uv)$$

$$= \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_2 - E_1} \mu_2(uv) + \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \vee \mu_2(uv)$$

$$+ \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv) - \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv)$$

$$= \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_2 - E_1} \mu_2(uv) + \sum_{uv \in E_1 \cap E_2} \mu_1(uv)$$

$$+ \sum_{uv \in E_1 \cap E_2} \mu_2(uv) - \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv)$$

$$\left[\text{Since} \sum \mu_1(u,v) + \sum \mu_2(u,v) = \sum \mu_1(u,v) \vee \mu_2(u,v) + \sum \mu_1(u,v) \wedge \mu_2(u,v)\right]$$

$$d_{G_1 \cup G_2}(u) = \sum_{uv \in E_1} \mu_1(uv) + \sum_{uv \in E_2} \mu_2(uv) - \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv)$$

$$\left[\because \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_1 \cap E_2} \mu_1(uv) = \sum_{uv \in E_1} \mu_1(uv)\right]$$

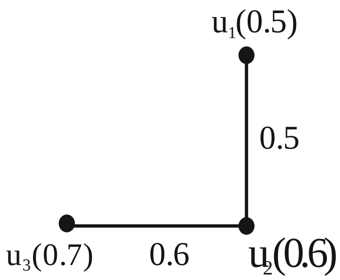
$$\therefore d_{G_1 \cup G_2}(u) = d_{G_1}(u) + d_{G_2}(u) - \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv)$$

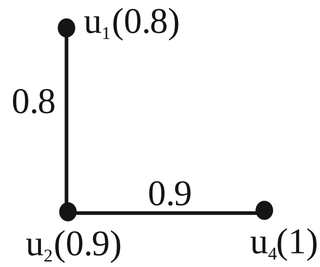
Hence the theorem.

7. Strong and Complete Nature in the Union of Two Fuzzy Graphs

The union of two strong fuzzy graphs need not be a strong fuzzy graph. It is illustrated in the following example.

Example 7.1.





G_1: (σ_1, μ_1) strong fuzzy graph $\qquad$ G_2: (σ_2, μ_2) strong fuzzy graph

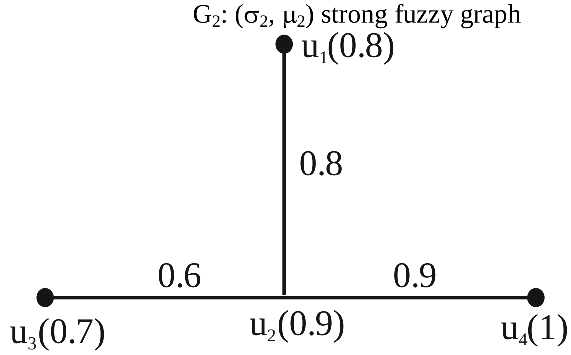


Figure 2. $G_1 \cup G_2$: (σ, μ) not a strong fuzzy graph

The following theorem shows that the conditions that can be included to make $G_1 \cup G_2$, a strong fuzzy graph.

Theorem 7.2. It G_1: (σ_1, μ_1) and G_2: (σ_2, μ_2) are two strong fuzzy graphs than $G_1 \cup G_2$: $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is again a strong fuzzy graph if $\sigma_1(u) \le \sigma_1(v)$ and $\sigma_2(u) \le \sigma_2(v)$ $\forall$ $u \in V_1 \cup V_2$ and $v \in V_1 \cap V_2$ and $\sigma_1 \le \mu_2$.

Proof:

Since $\sigma_1 \le \mu_2$

$\sigma_1 \le \mu_2 \le \sigma_2$

$\Rightarrow \sigma_1 \le \sigma_2$ $\qquad$ (1)

Since $\sigma_1 \le \mu_2$

$\mu_1 \le \sigma_1 \le \mu_2$

$\Rightarrow \mu_1 \le \mu_2$ $\qquad$ (2)

Case (1): If $uv \in E_1$, $v \in V_1 \cap V_2$ and $u \in V_1$

$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v)$

$= \sigma_1(u) \wedge \sigma_1(v)$

$= \sigma_1(u)$ $\qquad$ (3)

$(\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v) = \sigma_1(u) \wedge [\sigma_1(v) \vee \sigma_2(v)]$

$[\therefore\ \sigma_1 \cup \sigma_2(u) = \sigma_1(u)$ if $u \in V_1]$

$= [\sigma_1(u) \wedge \sigma_1(v)] \vee [\sigma_1(u) \wedge \sigma_2(v)]$

$= \sigma_1(u) \vee [\sigma_1(u) \wedge \sigma_2(v)]$ $\qquad$ $[\because\ \sigma_1(u) \le \sigma_1(v)]$

$= \sigma_1(u)$ $\qquad$ (4) $\qquad$ $[\because\ a \vee (a \wedge b) = a]$

From (3) and (4)

$(\mu_1 \cup \mu_2)(u, v) = (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$

Case 2: If $uv \in E_2$, $v \in V_1 \cap V_2$ and $u \in V_2$

$(\mu_1 \cup \mu_2)(u, v) = \mu_2(u, v)$

$= \sigma_2(u) \wedge \sigma_2(v)$

$= \sigma_2(u)$ $\qquad$ (5) $\qquad$ $[\because \sigma_2(u) \le \sigma_2(v)]$

$(\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$

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$= \sigma_2(u) \wedge [\sigma_1(v) \vee \sigma_2(v)]$ $\quad [\because (\sigma_1 \cup \sigma_2)(u) = \sigma_2(u), \text{ if } u \in V_2]$

$= [\sigma_2(u) \wedge \sigma_1(v)] \vee [\sigma_2(u) \wedge \sigma_2(v)]$

$= [\sigma_2(u) \wedge \sigma_1(v)] \vee \sigma_2(u)$

$[\because \sigma_2(u) \leq \sigma_2(v) \text{ if } v \in V_1 \cap V_2]$

$= \sigma_2(u)$ (6) $\quad [\because (a \wedge b) \vee a = a]$

From (5) and (6)

$(\mu_1 \cup \mu_2)(u, v) = (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$

Case 3: If $uv \in E_1 \cap E_2$ and $u, v \in V_1 \cap V_2$

$(\mu_1 \cup \mu_2)(u, v) = \mu_1(u, v) \vee \mu_2(u, v)$

$= \mu_2(u, v)$ $\quad [\because \mu_1 \leq \mu_2]$

$= \sigma_2(u) \wedge \sigma_2(v)$ $\quad$ (7)

$(\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$

$= [\sigma_1(u) \vee \sigma_2(v)] \wedge [\sigma_1(u) \vee \sigma_2(v)]$

$= \sigma_2(u) \wedge \sigma_2(v)$ (8) $\quad [\because \sigma_1 \leq \sigma_2]$

From (7) and (8)

$(\mu_1 \cup \mu_2)(u, v) = (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$

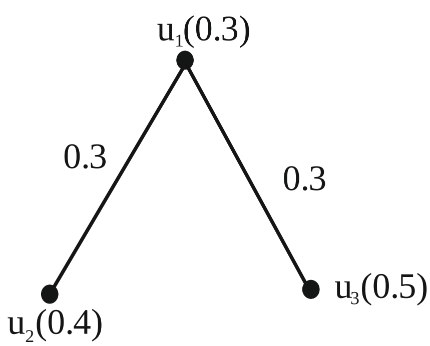
Hence the theorem ■

Remark 7.3. In general union of two complete fuzzy graph need not be a complete fuzzy graph it can be illustrated from the following example.

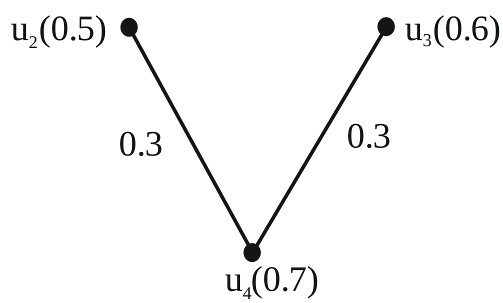
8. Regular Fuzzy Nature of the Union of Two Fuzzy Graphs

Union of two non regular fuzzy graphs can be a regular fuzzy graph. This can be illustrated from the following example.

Example 8.1.



$G_1: (\sigma_1, \mu_1)$
non regular fuzzy graph



$G_2: (\sigma_2, \mu_2)$
non regular fuzzy graph

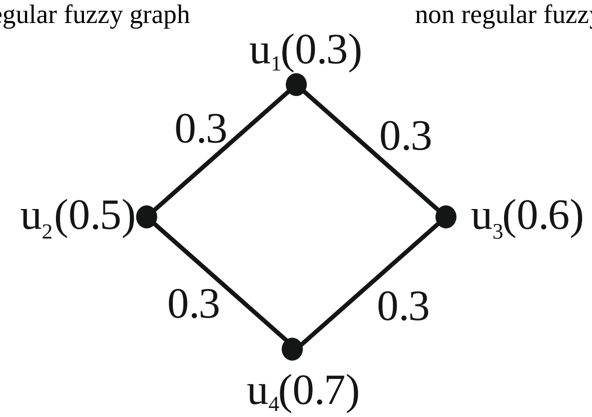


Figure 3. $G_1 \cup G_2: (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ regular fuzzy graph.

Union two regular fuzzy graph can be a non regular fuzzy graph. This result is illustrated with an example as follows.

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10. Conclusion

In this paper, we have obtained the relation between the order and size of a fuzzy graph with union of two fuzzy graphs. We find the degree of a vertex in the union of two fuzzy graphs. We have given one example for union of two complete fuzzy graph need not be a complete fuzzy graph. We have proved a theorem that the condition for the union of two strong fuzzy graphs is again a strong fuzzy graph. Also we have proved that the necessary and the sufficient condition for the union of two regular fuzzy graph to be a regular fuzzy graph.

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