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‘A BIRTH DEATH PROCESS IN STOCHASTIC PROCESS ’

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Introduction

In this paper, we investigate the Birth Death Chain, which is an important sub-class of Markov Chain. In the Birth Death Chain, the possible changes consists of an increase or decrease by one, or no change. Thus, the state space of the Birth Death Chain is the set of non- negative integers. The Birth Death Chains are frequently used to model growth of biological populations. The variety of dynamic behavior exhibited by many species of plants, insects and animals has stimulated great interest in the development of mathematical models. In many ecological problems, such as animals populations, epidemics and competition between species, their patterns of growth are influenced by population size.

The Birth Death Chain is not only useful for studying processes of biological populations, it is also used model the states of chemical systems. For example, the radioactive transformations can be modeled as a Birth process. In the radioactive transformation, the radioactive atoms are unstable and disintegrate stochastically. The new atoms are also unstable and could emit radioactive particles. Then these new atoms will decay with specified rates from one state to the adjacent state. This process can be modeled by the Birth Death Chain.

The queuing model is another important application of the Birth Death Chain in a wide range of areas, such as computer networks and telecommunications. For example, the queuing can be used to optimize the size of the storage space, to determine the trade-off between throughput and inventory, and to exhibit the propagation of blockage.

We consider the Poisson process and saw that it can be used to describe the arrivals of service requests in many cases of great practical interest. In a practical queuing system, the request arrivals result in resource allocation and eventually the users get served and leave the queue. It is customary to view this process as a member of a wider class of stochastic processes that are commonly referred to as the Birth and Death. Within this framework, every incoming request is regarded as a Birth and every user that, after being served, leaves the system is regarded as a Death. For the Poisson process the average Birth rate is specified by the distribution parameter λ. The Birth rate can change as a function of the state of the queuing system. However, we can still say that in a short time interval h, the probability of a single Birth is equal to $\lambda_n h+o(h)$, where subscript n indicates one of the system states. Likewise, it is reasonable to assume that in a short time interval h, the number of users leaving the system is equal to $\mu_n h+o(h)$, where μ_n indicates the average Death rate, and index n referred to the state of queuing system. The Birth and Death process is frequently used as a mathematical model of a queuing system. The framework of the Birth and Death process will allow us to derive some results that describe the behavior of the queuing system in general.

Preliminaries

Poisson Process

A stochastic process $\{N(t), t \geq 0\}$ is said to be a Poisson process with intensity or rate $\lambda > 0$ if the following conditions are satisfied

1. It starts from 0, (i.e) N(0) = 0.
2. It has stationary and independent increments. Stationary means that for time points s and t, s>t, the probability distribution of any increment X_s-X_t depends only on the length of the intervals on equally long time intervals are identically distributed. Independent means that for non-overlapping intervals [t,s] and [u,v] the random variables X_s - X_t and $X_v - X_u$ are independent.
3. For every t > 0, N(t) has a Poisson distribution with parameter λ_t

Birth Death Process

Consider a stochastic process N(t) that is continuous in time but has a discrete state space $\Omega=\{0,1,2,\ldots.\}$. Suppose that this process describes a physical system that is in state E_n, n=0,1,2,… at time t, if and only if N(t)=n. then the system is described by the BIRTH AND DEATH PROCESS if there exists non-negative birth rate λ_n, n =0,1,2,…, and non negative death rates μ_n, n=0,1,2,…, such that the following postulates (sometimes called nearest neighbor assumptions) are true:

1. State changes are only allowed between state E_n to state E_{n+1} or from state E_n to E_{n-1} if n≥1, but from state E_n to state E_1 only.
2. If at time t the system is in state E_n, the probability that between time t and time t+h a transition from state E_n to state E_{n+1} occurs equals $\lambda_n h+o(h)$, and the probability of transition from E_n to E_{n-1} is $\mu_n h+o(h)$,(n≥1).

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Stochastic Process

Let T be a set, and $t\epsilon T$ a parameter, in this case signifying time. Let X(t) be a random variable $\forall t\epsilon T$. then the set of random variables {X(t), $t\epsilon T$} is called a stochastic process

Arrival Pattern

In queuing the arrival process is usually stochastic. As a result it is necessary to determine the probability distribution of the inter-arrival times (times between successive customer arrivals) as well. Also customers can arrive in individually or simultaneously (batch or bulk arrivals)

Service Pattern

As in arrivals, a probability distribution is needed for describing a sequence of customer service time. Service may also be single or batch. The service process may depend on the number of customers waiting in queue for service. In this it is called state depended service

Queue Discipline

Queue discipline refers to the manner in which customers are selected for service when a queue has formed. The default is FCFS (i,e) first come, first served. Some others are LCFS (i,e) last come, first served. RSS(Random Service Selection) (i,e) selection for service in random order independent of the time of arrivals and there are other priority systems where customers are given priorities upon entering the system , ones with higher priority are selected.

Random Variable

A single-valued real function X(e) defined on the set E of elementary events is called Random Variables if inverse image of every interval I and the real axis of the form $(-\infty,x)$ is random event.

Basic Queuing Theory

Definition 1.1

Consider a stochastic process N(t) that is continuous in time but has a discrete state space $\Omega=\{0,1,2,\ldots.\}$. Suppose that this process describes a physical system that is in state E_n, n=0,1,2,… at time t, if and only if N(t)=n. then the system is described by the BIRTH AND DEATH PROCESS if there exists non-negative birth rate λ_n, n =0,1,2,…, and non negative death rates μ_n, n=0,1,2,…, such that the following postulates (sometimes called nearest neighbor assumptions) are true:

1. State changes are only allowed between state E_n to state E_{n+1} or from state E_n to E_{n-1} if $n\geq1$, but from state E_n to state E_1 only.
2. If at time t the system is in state E_n, the probability that between time t and time t+h a transition from state E_n to state E_{n+1} occurs equals $\lambda_n h+o(h)$, and the probability of transition from E_n to E_{n-1} is $\mu_n h+o(h),(n\geq1)$.

Characteristics 1.2

A queueing system is usually described by five basic characteristics of queueing processes:

1. Arrival pattern of customers.
2. Service pattern of customers.
3. Queue discipline .
4. System capacity .
5. Number of service channels.
6.

In most queueing systems, the arrival pattern will be stochastic. We therefore wish to know the probability distribution describing the inter arrival times (times between successive customer arrivals), and also whether customers arrive singly or in groups. If an arrival pattern does not depends on time (in other words, the probability distribution that describes the arrival process is time-independent), it is called a **stationary** arrival pattern. An arrival pattern that is time-dependent is called **non-stationary.**

Queue Discipline 1.3

Queue discipline indicates the manner in which customers receive service. The most common discipline are

1. FCFS (FIRST COME FIRST SERVED)
2. LCFS (LAST COME FIRST SERVED)

Or disciplines involving a set of priorities.

Notations to Prove the Properties of Birth Death Chain 1.4

First we define the transition probabilities of birth death chain as follows:

$$p_{ij} = \begin{cases} p_i & \text{if } j=i+1 \\ q_i & \text{if } j=i-1 \\ 1-p_i-q_i & \text{if } j=I \\ 0 & \text{else} \end{cases}$$

where p_i, $q_i \geq 0$ and $p_i + q_i \leq 0$. p_i is called the birth rate, and q_i is called the death rate. To ensure the zero boundary condition, we usually require $q_0 = 0$ and $p_0 > 0$. Then the state space is S = {0,1,2,…}.

To illustrate the basic setup, we use a simple queuing model. Let X_n represent the number of people in a line for some service at time n. We assume that people arrive at a rate of λ. Then we have $p_i = \lambda$ for each $i \in S$.

If there is a single server that serves people at a rate of μ, then we have $q_i = \mu$ for each $i \in S$ expect $q_0 = 0$. If there are k>1 servers and each servers people at a rate μ, then we have the following configuration:

$$q_i = \begin{cases} i\mu, & \text{if } i \leq k \\ k\mu, & \text{if } i \geq k \end{cases}$$

Theorem 1.5

The Birth Death Chain is transient under [1,2] if and only if

$$\sum_{k=-}^{n} \frac{q_i, \ldots\ldots q_k}{p_i, \ldots\ldots p_k} < \infty$$

Proof

Let α_n denote the probability that the chain, starting at state n ϵ {0,1,2,….}, ever returns to state 0. Then we have

$$\alpha_n = P\{X_i = 0 \text{ for some } i \geq 1 | X_0 = n\}$$
$$= \sum_k P\{X_i = 0 \text{ for some } i \geq 1 | X_1 = k\} P\{X_i = k | X_0 = n\}$$
$$= p_n \alpha_{n+1} + q_n \alpha_{n-1} + (1-p_n-q_n)\alpha_n$$

Which yields the relation as follows?

$$(p_n + q_n)\alpha_n = p_n \alpha_{n+1} + q_n \alpha_{n-1}$$

Then we have inductive relation as follows:

$$\alpha_n - \alpha_{n+1} = \frac{q_n}{p_n}(\alpha_{n-1} - \alpha_n)$$

Iterating the above equation yields

$$\alpha_n - \alpha_{n+1} = \frac{q_1, \ldots q_n}{p_1, \ldots\ldots p_n}(\alpha_0 - \alpha_1)$$

Finally, we have

$$\alpha_{n+1} = (\alpha_1 - 1)\sum_{k=1}^{n} \frac{q_1, \ldots q_k}{p_1, \ldots p_k} + 1$$

If the following term convergence

$$\sum_{k=-}^{n} \frac{q_i, \ldots\ldots q_k}{p_i, \ldots\ldots p_k} < \infty$$

Then we let

$$\alpha_1 = \left(\sum_{k=1}^{\infty} \frac{q_1, \ldots q_k}{p_1, \ldots\ldots p_k}\right) / \left(\sum_{k=0}^{\infty} \frac{q_1, \ldots q_k}{p_1, \ldots\ldots p_k}\right)$$

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So that

$$\alpha_{n+1} = \left(\frac{1}{\sum_{k=0}^{\infty} \frac{q_1,\ldots q_k}{p_1,\ldots\ldots p_k}}\right) \sum_{k=n+1}^{\infty} \frac{q_1,\ldots q_k}{p_1,\ldots\ldots p_k} \to 0$$

as n → ∞

Let us continue consider the Queueing model with single server. Then we have

$$\sum_{k=1}^{\infty} \frac{q_1,\ldots\ldots q_k}{p_1,\ldots\ldots p_k} = \sum_{k=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^k$$

Which converges if $\mu<\lambda$. This implies that if the arrival rate is strictly greater than the leaving rate, then the server queue is transient.

The Expected Extinction Time 1.6

Consider the Birth Death Model with 0 as a recurrent absorbing state ($p_0=0$). Then the population will die out at some point. Let T_k denote the expected time before the population hits zero, conditioned on an initial population of size k. then we have

$$T_k = p_k(1+T_{k+1})+q_k(1+T_{k-1})(1-p_k-q_k)(1+T_k)$$

After rearranging the equation, we have

$$T_{k+1} = T_k + \frac{q_k}{p_k}\left(T_k - T_{k-1} - \frac{1}{q_k}\right)$$

Since $T_0=0$, we have,

$$T_2 = T_1 + \frac{q_1}{p_1}\left(T_1 - \frac{1}{q_1}\right)$$

Iterating the above equations yields

$$T_m = T_1 + \sum_{k=1}^{m-1} \frac{q_1,\ldots q_k}{p_1,\ldots\ldots p_k}\left(T_1 - \frac{1}{q_1} - \sum_{i=2}^{k} \frac{p_1,\ldots p_{i-1}}{q_1,\ldots\ldots q_i}\right)$$

It remains to determine the exact value of T_1. To determine T_1, we modify the model slightly. Let $p_0=1$ and T_0 denote the first return time. Then $E(T_0)=T_1+1$, where T_1 is what we want. Let π be stationary distribution of the modified model, then we have

$$\pi_0 = \frac{1}{1+\sum_{k=1}^{\infty} \frac{p_0,\ldots p_{k-1}}{q_1,\ldots\ldots q_k}}$$

thus we have

$$T_1 = E(T_0) - 1 = \frac{1}{\pi_0} - 1 = \frac{1}{q_1} + \sum_{k=2}^{\infty} \frac{p_1,\ldots\ldots p_{k-1}}{q_1,\ldots\ldots q_k}$$

Finally, we can get

$$T_m = T_1 + m - 1\sum_{k=1} \left(\frac{q_1,\ldots\ldots q_k}{p_1,\ldots\ldots p_k} \sum_{i=k+1}^{\infty} \frac{p_1,\ldots\ldots p_{i-1}}{q_1,\ldots\ldots q_i}\right)$$

Continuous and Discrete Type Birth Death Process

Definition 2.1

Let T be a set, and $t\in T$ a parameter, in this case signifying time. Let X(t) be a random variable $\forall t\in T$. then the set of random variables $\{X(t), t\in T\}$ is called a stochastic process

Definition 2.2

Let X(t) to be the state of the stochastic process at time t. If T is countable, for example, if we let t=0,1,2,…, then we say that $\{X(t),t\in T\}$ is said to be a discrete time process. If, on the other hand, we let T be an interval of $[0,\infty)$, then the stochastic process is said to be a continuous-time process. The set of values of X(t) is called the state space, which can also be either discrete (finite or countably infinite), or continuous(a subset of R or R^n).

Definition 2.3

A stochastic process {X(n), n∈N} is called a Markov chain if, for all times n∈N and for all states $(i_0, i_1, \ldots, i_n)$

(2.1) $P=\{X_n = i_n \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}\} = P\{X_n = i_n \mid X_{n-1} = i_{n-1}\}$

equation(2.1) is called the Markov property, and in fact, any stochastic process satisfying the Markov property will be a Markov chain, whether it is discrete- time (as we defined above), or continuous-time process

We call conditional probability

$P\{X_n=j|X_{n-1}=i\}, i,j \in S$

The transition probability from state i to state j , denoted by $p_{ij}(n)$

Definition 2.4

A Markov Chain is called time-homogeneous if $p_{ij}(n)$ does not depend on n. In other words,

$P\{X_n= j \mid X_{n-1}= i\} = P\{X_{n+m} = j \mid X_{n+m-1}= I\}$

For m ∈ N and m ≥ - (n—1).In the future, unless otherwise noted, all Markov chains will be assumed to be time homogeneous and we will devote the time probability from state i to state j by P_{ij}.

Given the transition probabilities, we can construct the transition matrix P for the Markov Chain. P is an N × N matrix where the (i,j) entry P_{ij} is p_{ij}.In order for a matrix to be the transition matrix for a Markov chain, it must be a stochastic matrix. In other words, it must satisfy the following two properties:

(2.2) $\leq P_{ij} \leq 1, \quad 1\leq i, j \leq N$

(2.3) $\sum_{j=1}^{N} P_{ij} = 1, \quad 1\leq i \leq N.$

Given a transition matrix P, an initial probability distribution ∅ where ∅(i) =P{X_o= i} , i = 1, ……,N, we can find the probabilities that the Markov chain will be in a certain state i at a given time n. we define the n-step probabilities $P_{i,j}^n$ as the following:

$P_{ij}^n=P\{X_n=j|X_0=i\}=P\{X_{n+k}=j|X_k=i\}$

The latter part of the equation follows from time-homogeneity. Then we have

(2.4) $P\{X_n=j\}=\sum_{i\in S} ¤(i)\, p_n(i,j) = \sum_{i\in S} ¤(i) P\{X_n = j \mid X_0 = i\}$

Where S is the state space.

Theorem 2.5

The n-step transition probability $p_n(i , j)$ is actually the (i , j) entry in the matrix P^n

Proof

We will prove this by induction. Let n=1. Then, by the definition of the transition matrix P, p_{ij} is the (i,j) entry and our theorem holds.

Now, assume that it is true for a given n. Then

$$P_{ij}^{n+1}=P\{X_{n+1}=j|X_0=i\}$$
$$=\sum_k P\{X_n=k|X_0=i\}P\{X_{n+1}=j|X_n=k\}$$
$$=\sum_k p_{ik}^n p_{kj}$$

But since p_{ik}^n is the (i,k) entry of P^n by assumption, the final sum is just the (i,j) entry of $P^nP=P^{n+1}$

The intial probability distribution can be written as a vector:

$\Phi_0=(\Phi_0(1),\ldots\ldots \Phi_0(N)).$

Then we can find the distribution at time n, $\Phi_n(i)=P\{X_n=i\}$; and $\Phi_n= \Phi_0P^n$.

Definition 2.6

Two states i and j of a Markov chain communicate iff there exists m,n ≥ 0 such that $p_m(i,j) >0$ and $p_n(i,j) > 0$

Definition 2.7.
A Martkov chain is irreducible if ∀i,j
$\exists n = n\ (i,j)$ with $p_n(i,j)>0$
Simply put, a Markov chain is irreducible if it has only one communication class.

Poisson Process
Consider a stochastic process that counts arrivals, $\{N(t),\ t \geq 0\}$ where N(t) denotes the total number of arrivals up to time t, and N(0) =0. The stochastic process N(t) is considered a Poisson process N(t) is considered a Poisson process with rate parameter λ if it satisfies the following three conditions :

1. The probability that an arrival occurs between time t and t+Δt is λΔt+ o(Δt), where λ is a constant independent of N(t). we introduce the notation o(Δt) to denote a function of Δt satisfying
$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$
2. The probability that mpore than one arrival occurs between t and t+Δt is o(Δt).
3. The numbers of arrivals in non-overlapping intervals are statistically independent, so that the process has independent increments.

Let $p_n(t)$ be the probability of n arrivals in a time intervals of length t, where $n \in N \cup 0$. Now,
$p_n(t+\Delta t)$ =P {n arrivals in t and none in Δt}
P{n-1 arrivals in t and one in Δt}
+……+P{no arrivals in t and n in Δt}
Using assumptions (i),(ii),(iii), we have
(2.5) $p_n(t+\Delta t)= p_n(t)[1- \lambda\Delta t- o(\Delta t)] + p_{n-1}(t)[\lambda\Delta t+o(\Delta t)]+o(\Delta t)$,
Where the last o(Δt) represents the terms P{n-j arrivals in t and j in Δt} where $2 \leq j \leq n$ For n=0, we have
(2.6) $p_o(t+\Delta t)= p_o(t)[1-\lambda\Delta t-o(\Delta t)]$.
From equation (2. 5) and (2.6) and combining all the o(Δt) terms, we have
(2.7) $p_o(t+\Delta t)-p_o(t) = -\lambda\Delta t p_o(t)+ o(\Delta t)$
And
(2.8) $p_n(t+\Delta t)-p_n(t) = -\lambda\Delta t p_n(t)+\lambda\Delta t p_{n-1}(t)+o(\Delta t)$.
From equation (2.7) and (2.8) we can take the limit as $\Delta t \to 0$ and get the differential – difference equations

(2.9) $$\lim_{\Delta t \to 0}\left[\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t} = -\lambda\, p_0(t) + \frac{o(\Delta t)}{\Delta t}\right]$$

(2.10) $$\lim_{\Delta t \to 0}\left[\frac{p_n(t+\Delta t)-p_n(t)}{\Delta t} = -\lambda\, p_n(t) + \lambda\, p_{n-1}(t)\frac{o(\Delta t)}{\Delta t}\right],$$

With simplify to

(2.11) $$\frac{dp_0(t)}{dt} = -\lambda\, p_0(t)$$ and

(2.12) $$\frac{dp_n(t)}{dt} = -\lambda\, p_n(t) + \lambda\, p_{n-1}(t) \qquad (n \geq 1)$$

Now we have an infinite set of linear , first order ordinary differential equations to solve. Equation (2.11) has the general solution $p_o= Ce^{-M}$, where C=1 since p_o=0.

Now , let n=1 in (2.12) so that
$$\frac{dp_1(t)}{dt} = -\lambda\, p_1(t) + \lambda\, p_0(t)$$
Or
$$\frac{dp_1(t)}{dt} + \lambda\, p_1(t) = \lambda\, p_0(t) = \lambda\, e^{-\lambda t}.$$
Solving this equation we get
$$p_1(t) = Ce^{-\lambda t} + \lambda te^{-\lambda t}$$
Since $p_n(0)$ is 0 ∀ n> 0 we have C =0, and
$$p_1(t) = \lambda te^{-\lambda t}$$
Continuing this process for n = 2 and n = 3 we get

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$$p_2(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t}$$ and

$$p_3(t) = \frac{(\lambda t)^3}{3!} e^{-\lambda t} .$$

Continuous – Time Markov Chains 2.11

Let { X (t) , t ∈ T} be a continuous – time Markov chain. This means

$$T = \{t: 0 \le t < \infty\}$$

Consider any times s > t > u ≥ 0 and states i , j ; then

(2.13) $$p_{ij}(u,s) = \sum_r p_{ir}(u,t)\, p_{rj}(t,s)$$

Where $p_{ij}(u,s)$ is the probability of moving from state i to state j in the time beginning at u and ending at s, and the summation is over the entire state space of the chain.

Letting u= 0 and s= t+Δt gives

(2.14) $$p_{ij}(0, t+\Delta t) = \sum_r p_{ir}(0,t)\, p_{rj}(t, t+\Delta t)$$

Let $p_i(0)$ be the probability that the chain starts in state i at time 0 and $p_j(t)$ be the probability that the chain is in state j at time t regardless of starting state ,

We multiply (2.14) by $p_i(0)$ and sum over all states to get

$$\sum_i p_i(0)\, p_{ij}(0, t+\Delta t) = \sum_r \sum_i p_{ir}(0,t)\, p_i(0)\, p_{rj}(t, t+\Delta t), \text{ Or} \qquad (2.15)$$

$$p_j(t+\Delta t) = \sum_r p_r(t)\, p_{rj}(t, t+\Delta t).$$

For the Poisson process

$$P_{rj}(t, t+\Delta t) = 1-\lambda\Delta t + o(\Delta t) \quad \begin{cases} \lambda\Delta t + o(\Delta t) & \text{if } r = j-1, j \ge 1, \\ & \text{if } r = j, \\ o(\Delta t) & \text{lsewhere.} \end{cases}$$

Substituting this into (2.15) , we get

$$p_j(t+\Delta t) = [\lambda\Delta t + o(\Delta t)]\, p_{j-1}(t) + [1-\lambda\Delta t + o(\Delta t)]\, p_j(t) + o(\Delta t) \qquad (j \ge 1)$$

Which is (2.5).

Now if the transistion probability functions p(u,s) have continuous functions $q_i(t)$ and $q_{ij}(t)$ associated with them so that

(2.16) P{a change of state in (t, t+Δt)} = 1- p_{ii} (t, t+Δt) = q_i (t)Δt +o(Δt)

And

(2.17) p_{ij} (t, t+Δt) = q_{ij}(t) Δt +o(Δt),

By taking equation (2.13) and using (2.16) and (2.17) we get the Kolmogorov forward and backwards equations

(2.18) $$\frac{\partial}{\partial t} p_{ij}(u,t) = -q_j(t)\, p_{ij}(u,t) + \sum_{r \ne j} p_{ir}(u,t)\, q_{rj}(t)$$

(2.19) $$\frac{\partial}{\partial u} p_{ij}(u,t) = q_i(u)\, p_{ij}(u,t) + \sum_{r \ne i} q_{ir}(u)\, p_{rj}(u,t) .$$

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