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MAXIMUM PRINCIPLE AND EXISTENCE RESULTS FOR NONLINEAR ELLIPTIC SYSTEMS ON R^N

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Abstract
In this work we give necessary and sufficient conditions for having a maximum principle for cooperative elliptic systems involving p- Laplacian operator on the whole $\mathbb{R}^N$. This principle is then to yield solvability for the considered cooperative elliptic system by an approximation method.
Keywords: Maximum Principle , p- Laplacian Operator, Elliptic System, Approximation method.

1. Introduction

This work is concerned with the general nonlinear cooperative elliptic system

$$\begin{cases} -\Delta_p u = a\, m(x) |u|^{p-2} u + b m_1(x) h(u,v) + f & \text{in } R^N \\ -\Delta_q u = d\, n(x) |v|^{q-2} v + c n_1(x) k(u,v) + g & \text{in } R^N \\ u(x) \to 0,\ v(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \quad (1.1)$$

Here $\Delta_p u := \text{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $1< p < +\infty$, is the P- Laplacian operator. The parameters a, b, c, d are nonnegative real parameter. The functions h, k: $\mathbb{R}^2 \to \mathbb{R}$ are continuous and have some properties like the weight functions m, m_1, n, n_1 which will be specified laterThe aim of this work is to construct a Maximum Principle with inverse positivity assumptions which means that if f, g are nonnegative functions almost everywhere in R^N, then any solution (u, v) of (1.1) obey $u \geq 0$; $v\geq 0$ a.e in R^N.It is well known that the maximum principle plays an important role in the theory on nonlinear equations. For instance it is used to access existence results of solutions for linear and nonlinear differential equations. [1-15]. Most of the work deals with Maximum Principle for a certain class of functions h and k. This work deals with a more general class of functions h, k. For specific interest for our purposes is the work in [7] where a study of problems such as (1.1) was carried out in case of Ω in the presence of the weights m, m_1, n, n_1 with the particular case of h (s, t) and k(s,t).Clearly, our work extends the work [7] first by considering a problem with weights and next by dealing with a more general class of function h, k in the of whole of R^N. For instance this result can apply for the case.

$$h(s,t) = \begin{cases} s^{\alpha+1} |arctant|^{\beta} e^{-\beta t} & \text{for } t \geq 0, s\in\mathbb{R} \\ |s|^{\alpha} |t|^{\beta} t & \text{for } t \leq 0, s\in\mathbb{R} \end{cases}$$

$$k(s,t) = \begin{cases} s^{\alpha+1} e^{-us} |arctant|^{\beta} & \text{for } s \geq 0, t\in\mathbb{R} \\ |s|^{\alpha} s |t|^{\beta} & \text{for } s \leq 0, t\in\mathbb{R} \end{cases}$$

which is not taking into account in [7].

The remainder of the work is organized as follows: In Preliminary Section 2 we specify the required assumptions on the data of our considered problem and we briefly give some known results relative to the principal positive eigenvalue of the p- Laplacian operator. In section 3, the Maximum principle for (1.1) is given and is shown to be proven full enough to yield existence results of solution for (1.1) in Section 4 by using a approximation method.

2. Preliminaries

Throughout this work assume that, $1 < p, q < n$ and

(B1) $\alpha, \beta \geq 0$; $b, c \geq 0$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{p} = 1$;

(B2) $f \geq 0$, $f\in L^{(p*)'}(R^N)$; $g \geq 0$, $g\in L^{(q*)'}(R^N)$ with $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = 1$;

(B3) m, m_1, n, n_1 are smooth weights such that m, n>0 $m\in L^{\infty}_{loc}(R^N) \cap L^{N/p}(R^N)$, $n\in L^{\infty}_{loc}(R^N) \cap L^{N/q}(R^N)$, and $0< m_1, n_1\leq m^{(\alpha+1)/p}\, n^{(\beta+1)/q}$ Here $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$ denote the critical Sobolev exponents of p and q respectively; p' is the Holder conjugate of p.

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(B4) The functions h and k satisfy the sign conditions:
$t.h(s,t) \geq 0$, $s.k(s,t) \geq 0$ for $(s,t) \in \mathbb{R}^2$ and there exist $\Gamma > 0$ such that
$h(s,-t) \leq -h(s,t)$ for $t \geq 0$, $s \in \mathbb{R}$; $h(s,t) = \Gamma^{\alpha+\beta+2-q}|s|^{\alpha}|t|^{\beta}t$ for $t \leq 0$, $s \in \mathbb{R}$ $k(-s,t) \leq -k(s,t)$ for $s \geq 0$, $t \in \mathbb{R}$; $k(s,t) = \Gamma^{\alpha+\beta+2-q}|s|^{\alpha}s|t|^{\beta}$ for $s \leq 0$, $t \in \mathbb{R}$

We denote by $W_0^{1,p}(R^N)$ (with $1 < s < N$) the completion of $C_0^{\infty}(R^N)$ with respect to the norm $\|u\|_{W_0^{1,p}(R^N)} = \left(\int_{R^N}|\nabla u|^p\right)^{\frac{1}{p}}.W_0^{1,p}(R^N)$ is a reflexive Banach space and it can be shown [8] that $W_0^{1,p}(R^N) = \{u \in L^{p*}(R^N) : \nabla u \in (L^P(R^N))^N\}$

Here and henceforth the Lebesgue norm in $L^P(R^N)$ will be denoted by $||.||_p$ and the usual norm of $W_0^{1,p}(R^N)$ by $||.||$.The positive and negative part of a function u are defined respective as $u^+ = \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. Equalities (and inequalities) between two functions must be understood a.e. (R^N)

Consider the eigenvalue problem with weight g. For a given $g \in L^{N/P}(R^N) \cap L^{\infty}(R^N)$, g (x) a.e in R^N it was known that the eigenvalue problem.

$$\left.\begin{array}{ll} -\Delta_p u = \lambda_g(x)\,|\nabla u|^{p-2}u & \text{in } R^N \\ u(x) \to 0, \text{ as } |x| \to +\infty & \text{in } R^N \end{array}\right\} \qquad (2.1)$$

admits an unique positive first eigen value $\lambda_1(g, p)$ with a nonnegative eigenfunction.

Moreover, this eigenvalue is isolated, simple and as a consequence of its variational characterization one has $\lambda_1(g, p)\int_{R^N} g(x)|u|^p \leq \int_{R^N}|\nabla u|^p \ \forall u \in W_0^{1,p}(R^N)$

Now we denote by Φ (respectively Ψ) the positive eigenfunction associated with
$\lambda_1(m, p)$ (respectively $\lambda_1(n, q)$) normalized by $\int_{R^N} m(x)|\Phi|^p = 1$ (resp $\int_{R^N} n(x)|\Psi|^q = 1$).

The functions Φ and Ψ belong to $C^{1,\alpha}(R^N)$ and by the weak maximum principle,

$\frac{\partial \Phi}{\partial v} < 0$ and $\frac{\partial \Psi}{\partial v} < 0$ on R^N where v is the unit exterior normal.

3. Maximum Principle

We assume that $1 < p, q < N$ and also that hypothesis (B_3) is satisfied. We begin by consider the problem

$$\begin{array}{ll} -\Delta_p u = \mu m(x)|u|^{p-2}u + h(x) & \text{in } R^N \\ u(x) \to 0, & \text{as } |x| \to +\infty \end{array} \qquad (3.1)$$

The following result was proved in [9, 10]

Preposition 2.1. For all $r > 0$, any solution u of [2.1] belongs to $C^{1,\gamma}(B_r)$, where $\gamma = \gamma(r) \in]0,1[$ are B_r is the ball of radius r centred at the origin.

$$\text{Let } a_1(r) := \inf_{B_r} k_1(x) \text{ and } a_2(r) := \sup_{B_r} k_2(x) \qquad (3.2)$$

$$\text{where } k_1(x) := \left[\frac{n_1(x)}{n(x)}\right]^{\frac{\beta+1}{q}}\left[\frac{\Phi(x)^p}{\Psi(x)^q}\right]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}},\ k_2(x) := \left[\frac{m(x)}{m_1(x)}\right]^{\frac{\alpha+1}{q}}\left[\frac{\Phi(x)^p}{\Psi(x)^q}\right]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}},$$

We denote by $a_{1\infty} = \lim_{r\to+\infty} a_1(r)$ and $a_{2\infty} = \lim_{r\to+\infty} a_2(r)$. Let set $\Theta = \frac{a_{1\infty}}{a_{2\infty}}$. (3.3)

The following inequalities can easily be proved

$$\Theta = \frac{a_1(r)}{a_2(r)}.\ \text{for all } r > 0 \text{ and } 0 \leq \Theta \leq 1 \qquad (3.4)$$

We now turn to our first main result

A Maximum Principle holds for the system (1.1) if $f \geq 0$ and $g \geq 0$ implies $u \geq 0$ and $v \geq 0$ a.e in R^N.

By a solution (u, v) of (1.1), we mean a weak solution i.e., $(u, v) \in W_0^{1,p}(R^N) \times W_0^{1,q}(R^N)$ such that

$$\int_{R^N}|\nabla u|^{p-2}\nabla u.\nabla w = \int_{R^N}\left[am(x)|u|^{p-2}uw + bm_1(x)h(u,v)w + fw\right]$$

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$\int_{R^N} |\nabla|^{q-2} \nabla v . \nabla z = \int_{R^N} |$ [dn (x)| v $|^{q-2}$ vz+ cn_1 (x) k (u, v) z + gz]

for all (w, z) $\in W_0^{1,p}$ (R^N) x $W_0^{1,q}$ (R^N) . (3.5)

Note that by assumptions **(B1) – (B4),** the integrals in (3.5) are well – defined Regularity results from [12,13], weak solution (u,v) belong to $c^1(R^N)$ x $c^1(R^N)$. It is also know, that a weak of (1.1) decays to zero and infinity. Now ready to state the validity of the Maximum Principle for (1.1)

Theorem 3.1 Assume (B1) – (B4). Then the Maximum Principle holds for (1.1) if

(C_1) λ_1 (m, p) > a, **(C_2)** λ_1 (n, q) > d,

(C_3) λ_1 (m, p) –a$)^{(\alpha+1)/p}$ λ_1 (n, q) –d$)^{(\beta+1)/q}$> $b^{(\alpha+1)/p}$ $c^{(\beta+1)/q}$

Conversely if the Maximum Principle holds, then Conditions **(C_1) – (C_4)** are satisfied, where **(C_4)** (λ_1 (m, p) – a$)^{(\alpha+1)/p}$ λ_1 (n, q) –d$)^{(\beta+1)/q}$>Φ $b^{(\alpha+1)/p}$ $c^{(\beta+1)/q}$

Proof: The proof is party adapted from [1, 6] **Necessity Part:**

Assume that the Maximum Principle holds for system (1.1) If λ_1 (m, p) ≤ a then the functions f : (a-λ_1 (m, p) m (x) Φ^{p-1} and g : = 0 are nonnegative, however (-Φ, 0) satisfies (1.1), which contradicts the Maximum Principle. Similarly, if λ_1 (n, q) ≤ d then f : = 0 and g : = (d- λ_1 (n, q)) n (x) $\Psi^{q-1}$are nonnegative functions and (0, -Ψ) satisfies (1.1), which is a contradiction with the Maximum Principle.

Now, assume that λ_1 (m, p) > a, λ_1 (n, q) > d, and that **(C4)** does not hold; that is, (C4$'$) λ_1 (m, p) –a) $^{(\alpha+1)/p}$ λ_1 (n, p) –d) $^{(\beta+1)/q}$ <Θ b $^{(\alpha+1)/p}$ C $^{(\beta+1)/p}$

Set A = $\left(\frac{\lambda_1 (m,p)-a}{b}\right)^{(\alpha+1)1/p}$, B = $\left(\frac{\lambda_1 (n,p)-a}{c}\right)^{(\beta+1)1/q}$

Then by (C4$'$) AB ≤Θ which implies

A≤ $\frac{\Theta_2}{B}$, where Θ_1= $\inf_{R^N} k_1$ (x), Θ_2= $\sup_{R^N} k_2$ (x) (3.6)

Hence these exists $\xi\in R_+^*$ such that A $a_{2\infty}$ ≤ ξ≤ (1/B) $a_{1\infty}$. Let c_1, $c_2$be two positive real numbers such that ξ = $\left(\frac{c_2^q \Gamma^q}{c_1^p \Gamma^p}\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$. Using (3.6), (B1) and the above expression of ξ we have

[λ_1 (m, p) -a] m (x) [c_1Φ (x)$]^{p-1}$≤ $\Gamma^{\alpha+\beta+2-p}bm_1$ (x) [c_1 Φ (x)$]^\alpha$ [c_2 Ψ (x)$]^{\beta+1}$ for all x ∈ R^N [λ_1 (n, p) -d] n (x) [c_2Ψ (x) $]^{q-1}$≤ $\Gamma^{\alpha+\beta+2-q}$ cn_1 (x) [c_1 Φ (x)$]^{\alpha+1}$[c_2 Ψ (x)$]^\beta$ for all x ∈ R^N

Furthermore, using the inequalities in **(B4),** we obtain

-[λ_1 (m, p) –a] m (x) [c_1Φ (x)$]^{p-1}$ –bm_1 (x) h(-c_1 Φ, -c_2 Ψ) ≥0 for all x ∈ R^N and

-[λ_1 (n, q) –d] n (x) [c_2Ψ (x) $]^{q-1}$- cn_1 (x) k(-c_1 Φ, -c_2 Ψ) ≥0 for all x ∈ R^N

Hence 0≤-[λ_1 (m, p) –a] m (x) [c_1Φ (x)$]^{p-1}$ -bm_1 (x) h(-c_1 Φ, -c_2 Ψ) = f, for all x ∈ R^N.

0≤- [λ_1 (n, q) –d] n (x) [c_2Ψ (x)$]^{q-1}$ - cn_1 (x) k(-c_1 Φ, -c_2 Ψ)= g, for all x ∈ R^N.

are nonnegative functions and (-c_1 Φ ,-c_2 Ψ) is a solution of (1.1). This is a contradiction with the Maximum Principle.

Sufficiency Part : Assume that the conditions **(C1) – (C3)** are satisfied.

So for f ≥ 0, g ≥ 0, suppose that there exists a solution (u, v) of system (1.1).

Multiplying the first equation in (1.1) by u^- and the second on by v^- and integrating over R^N we have,

$$\int_{R^N} | \nabla u^- |^p = a \int_{R^N} m (x) | u^- |^p –b \int_{R^N} m_1 (x) h (u, v) u^- - \int_{R^N} f u^-$$

$$\int_{R^N} |\nabla v^- |^q = d \int_{R^N} n (x) | v^- |^q –c \int_{R^N} n_1 (x) k (u, v) v^- - \int_{R^N} g v^-$$

Then, using **(B4),** $\int_{R^N} | \nabla u^- |^p \le a \int_{R^N} m (x) | u^- |^p b \int_{R^N} m_1 (x) h (u, -v^-) u^-$

$$\int_{R^N} | \nabla v^-|^q \le d \int_{R^N} n (x) | v^-|^q –c \int_{R^N} n_1 (x) k (-u^- v) v^-$$

h $(u,-v^-)\, u^- = -\Gamma^{\alpha+\beta+2-p}(u^-)^{\alpha+1}(v^-)^{\beta+1}$, k $(-u, v)\, v^- = -\Gamma^{\alpha+\beta+2-p}(u^-)^{-\alpha+1}(v^-)^{\beta+1}$ and hence $\int_{R^N}|\nabla u^-|^p \le a\int_{R^N} m\,|u^-|^p + b\,\Gamma^{\alpha+\beta+2-p}\int_{R^N} m_1(x)(u^-)^{\alpha+1}(v^-)^{\beta+1}$

$$\int_{R^N}|\nabla v^-|^q \le d\int_{R^N} n\,|v^-|^q \le + c\,\Gamma^{\alpha+\beta+2-q}\int_{R^N} n_1(x)(u^-)^{\alpha+1}(v^-)^{\beta+1}$$

Combining the variational characterization of λ_1 (m, p) and λ_1 (n, q) with the Holder inequality and assumption **(B3),** we have

$$(\lambda_1(m,p)-a)\int_{R^N} m(x)|u^-|^p \le b\Gamma^{\alpha+\beta+2-p}\left(\int_{R^N} m(x)|u^-|^p\right)^{(\alpha+1)/q}\left(\int_{R^N} n(x)|v^-|^q\right)^{(\beta+1)/p}$$

$$(\lambda_1(n,q)-d)\int_{R^N} n(x)|v^-|^q \le c\Gamma^{\alpha+\beta+2-q}\left(\int_{R^N} m(x)|u^-|^p\right)^{(\alpha+1)/q}\left(\int_{R^N} n(x)|v^-|^q\right)^{(\beta+1)/p}$$

which implies

$$\left(\int_{R^N} m(x)|u^-|^p\right)^{(\alpha+1)/p}\left[(\lambda_1(m,p)-a)\left(\int_{R^N} m(x)|u^-|^p\right)^{(\beta+1)/q} - b\Gamma^{\alpha+\beta+2-p}\left(\int_{R^N} n(x)|v^-|^q\right)^{(\beta+1)/q}\right] \le 0,$$

$$\left(\int_{R^N} n(x)|v^-|^q\right)^{(\beta+1)/q}\left[(\lambda_1(n,q)-d)\right]\left(\int_{R^N} n(x)|v^-|^q\right)^{(\alpha+1)/p} - c\Gamma^{\alpha+\beta+2-q}\left(\int_{R^N} m(x)|u^-|^p\right)^{(\alpha+1)/p} \le 0. \quad (3.7)$$

Let us show that $u^- = v^- = 0$

If $\int_{R^N} m(x)|u^-|^p = 0$ or $\int_{R^N} n(x)|v^-|^q == 0$ then, using the fact that m > o, n >0, and (3.7) we obtain $u^- = v^- = 0$, which implies that the Maximum Principle holds

- If $\int_{R^N} m(x)|u^-|^p \neq 0$ and $\int_{R^N} n(x)|v^-|^p \neq 0$, then we have

$$\lambda_1(m,p) - a)\left(\int_{R^N} m(x)|u^-|^p\right)^{(\beta+1)/q} \le b\Gamma^{\alpha+\beta+2-p}\left(\int_{R^N} n(x)|v^-|^q\right)^{(\beta+1)/q}$$

$\lambda_1(n,q)-d)\left(\int_{R^N} n(x)|v^-|^q\right)^{(\alpha+1)/p} \le c\Gamma^{\alpha+\beta+2-q}\left(\int_{R^N} m(x)|u^-|^p\right)^{(\alpha+1)/p}$ which implies

$$(\lambda_1(m,p)-a)^{(\alpha+1)/p}\left(\int_{R^N} m(x)|u^-|^p\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le -b^{(\alpha+1)/p}\,\Gamma^{\alpha+\beta+2-p\,(\alpha+1)/p}\left(\int_{R^N} n(x)|v^-|^q\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$$

$$(\lambda_1(n,q)-d)^{(\beta+1)/q}\left(\int_{R^N} n(x)|v^-|^q\right)^{\frac{u+1}{p}\frac{\beta+1}{q}} \le -c^{(\beta+1)/q}\,\Gamma^{\alpha+\beta+2-q\,(\beta+1)/q}\left(\int_{R^N} m(x)|u^-|^p\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$$

Multiplying the two inequalities above and using the fact that

$$(\alpha+\beta+2-p)\frac{\alpha+1}{p} + (\propto+\beta+2-qs)\frac{\beta+1}{q} = (\alpha+\beta+2)\left(\frac{\alpha+1}{p}+\frac{\beta+1}{q}\right) - (\alpha+1) - (\beta+1) = 0 \quad (3.8)$$

$$(\lambda_1(m,p)-a)^{\frac{\alpha+1}{p}}(\lambda_1(n,q)-d)\, x\left(\left(\int_{R^N} m(x)|u^-|^p\right)\left(\int_{R^N} n(x)|v^-|^q\right)\right)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$$

$\le b^{\frac{u+1}{p}}\, c^{\frac{\beta+1}{q}}\left(\left(\int_{R^N} m(x)|u^-|^p\right)\left(\int_{R^N} n(x)|v^-|^q\right)\right)^{\frac{u+1}{p}\frac{\beta+1}{q}}$ and then

$$\left((\lambda_1(m,p)-a)^{\frac{u+1}{p}}(\lambda_1(n,q)-d)^{\frac{\beta+1}{q}} - b^{\frac{u+1}{p}}\,c^{\frac{\beta+1}{q}}\right) x\left(\left(\int_{R^N} m(x)|u^-|^p\right)\left(\int_{R^N} n(x)|v^-|^q\right)\right)^{\frac{u+1}{p}\frac{\beta+1}{q}} \le 0$$

Since **(C_1) – (C_3)** are satisfied and m, n > 0 the inequality above is not possible.

Consequently $u^- = v^- = 0$ and the Maximum Principle holds. □

4. Existence of solutions

Theorem 4.1 Assume **(B_1), (B_2), (C_1), (C_2), (C_3)** are satisfied. Then for $f \in L^{p^*}(R^N)$ and $g \in L^{q^*}(R^N)$, system (1.1) admits atleast one solution in $W_0^{1,p}(R^N) \times W_0^{1,q}(R^N)$.

The proof will be given in several steps and is partly adapted from [1,6,15]. To prove this theorem it requires the lemmas state below.We chooser r > 0 such that a + r > 0 and d + r > 0

Hence (1.1) reads as follows

$$\left.\begin{array}{l} -\Delta_p u + rm(x)|u|^{p-2}u = (a+r)\, m(x)|u|^{p-2}u + bn_1(x)\, h(u,v) + f \text{ in } R^N \\ -\Delta_q v + rn(x)|v|^{p-2}v = cn_1 k(u,v) + (d+r)n(x)|v|^{p-2}v + g \text{ in } R^N \\ u(x) \to 0,\ v(x) \to 0 \quad \text{as } |x| \to +\infty \end{array}\right\} \quad (4.1)$$

For $0 < \epsilon < 1$, now consider the system

$$\left.\begin{array}{l} -\Delta_p u_\epsilon + rm(x)|u_\epsilon|^{p-2}u_\epsilon = \hat{h}(x, u_\epsilon, v_\epsilon) + f \text{ in } R^N \\ -\Delta_q v_\epsilon + rn(x)|v_\epsilon|^{q-2}u_\epsilon = \hat{h}(x, u_\epsilon, v_\epsilon) + g \text{ in } R^N \\ u_\epsilon \to v_\epsilon \to 0 \quad \text{as } |x| \to +\infty \end{array}\right\} \quad (4.2)$$

where $\hat{h}(x, s, t) = (a+r)m(x)|s|^{p-2} s\,(1+\epsilon^{1/p}|s|^{p-1})^{-1} + bm_1(x)\, h(s,t)\,(1+\epsilon|h(s,t)|)^{-1}$,

$\hat{k}(x, s, t) = (d+r)\, n(x)|t|^{p-2}\, t\,(1+\epsilon^{1/q}|t|^{q-1})^{-1} + cn_1(x)k(s,t)\,(1+\epsilon|k(s,t)|)^{-1}$

Lemma 4.2

Under the hypothesis of theorem (3.1) System (4.2) has a solution in $W_0^{1,p}(R^N) \times W_0^{1,q}(R^N)$.

Proof : Let $\epsilon>0$ be fixed

- Construction of sub – solution and super – solution for system

$$\left.\begin{array}{l} -\Delta_p u + rm(x)|u|^{p-2}v = \hat{h}(x, u, v) + f \text{ in } R^N \\ -\Delta_q v + rn(x)|v|^{p-2}v = \hat{k}(x, u, v) + g \text{ in } R^N \\ u(x) \to 0,\ v(x) \to 0 \quad \text{as } |x| \to +\infty \end{array}\right\} \quad (4.3)$$

From **(B3),** the functions $\hat{h}$ and $\hat{k}$ are bounded ; that is, there exists a positive constant M such that $|\hat{h}(x, u, v)| < M$, $|\hat{K}(x, u, v)| < M$ $\quad \forall (u, v) \in W_0^{1,p}(R^N) \times W_0^{1,q}(R^N)$.

Let $u^0 \in W_0^{1,p}(R^N)$ (respectively $v^0 \in W_0^{1,q}(R^N)$) be a solution of $-\Delta_p u^0 + rm(x)|u^0|^{p-2}u^0 = M + f$ (respectively $-\Delta_p v^0 + rn(x)|v^0|^{q-2}v^0 = M + g$)

It was known that u_0 u^0, v_0, v^0 are exists, moreover we have

$-\Delta_p u_0 + rm(x)|u_0|^{p-2}u_0 - \hat{h}(x, u_0, v) - f \le 0 \qquad \forall v \in [v_0, v^0]$

$-\Delta_p u^0 + rm(x)|u^0|^{p-2}u^0 - \hat{h}(x, u^0, v) - f \ge 0 \qquad \forall v \in [v_0, v^0]$

$-\Delta_q v_0 + rn(x)|v_0|^{q-2}v_0 - \hat{k}(x, u, v_0) - g \le 0 \qquad \forall u \in [u_0, u^0]$

$-\Delta_q v^0 + rn(x)|v^0|^{q-2}v_0 - \hat{k}(x, u, v^0) - g \ge 0 \qquad \forall u \in [u_0, u^0]$

So (u_0, u^0) and (v_0, v^0) are sub – super solutions of (4.3)

- Let $K = [u_0, u^0] \times [v_0, v^0]$ and let $T : (u, v) \to (w, z)$ the operator such that

$$\left.\begin{array}{ll} -\Delta_q w + rm(x)|w|^{p-2}w = \hat{h}(x, u, v) + f & \text{in } R^N \\ -\Delta_q z + rn(x)|z|^{q-2}z = \hat{k}(x, u, v) + g & \text{in } R^N \\ w(x) \to 0,\ z(x) \to 0 \quad \text{as } |x| \to +\infty & \end{array}\right\} \quad (4.4)$$

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- Let us prove that $T(K) \subset K$. If $(u, v) \in K$, then

$$(-\Delta_p w-\Delta_p \xi^0) + rm(x)\left(|w|^{p-2}w - |\xi^0|^{p-2}\xi^0\right) = [\hat{h}(x, u, v) - M] \qquad (4.5)$$

Taking $(w - \xi^0)^+$ as test function in (4.5), we have

$\int_{R^N} (|\nabla w|^{p-2} \nabla w - |\nabla \xi^0|^{p-2} \nabla \xi^0) \nabla (w - \xi^0)^+ + r \int_{R^N} m(x)(|w|^{p-2} w - |\xi^0|^{p-2} \xi^0) \times (w - \xi^0)^+ = \int_{R^N} [h(x, u, v) - M)] (w - \xi^0)^+ \le 0$. Since the weight m is positive, by the monotonicity of the functions $s \to |s|^{p-2}$ and that of the p –Laplacian, we deduce that the last integral equal zero and the $(w - \xi^0)^+ = 0$. That is $w \le \xi^0$.Similarly we obtain $\xi^0 \le w$ by taking $(w - \xi^0)^-$ as test function in (4, 5). So we have $\xi_0 \le w \xi^0$ and $\eta_o \le z \le \eta^0$ and the step is complete.

- To show that T is completely continuous we need the following lemma.

Lemma 4.3 If $(u_n, v_n) \to (u, v)$ in $L^{p^*}(R^N) \times L^{q^*}(R^N)$ as $n \to \infty$ then

(1) $X_n = m(x)\dfrac{|u_n|^{p-2}u_n}{1+\epsilon^{1/p}|u_n|^{p-1}}$ Converges to $X = m(x)\dfrac{|u_n|^{p-2}u}{1+\epsilon^{1/p}|u|^{p-1}}$ in $L^{p*'}(R^N)$ as $n \to \infty$.

where $p^* = \dfrac{pN}{N(p-1)+p}$

(2) $Y_n = m_1(x)\dfrac{h(u_n v_n)}{1+\epsilon\, h(u_n v_n)}$ Converges to $Y = m_1(x)\dfrac{h(u,v)}{1+\epsilon\, h(u,v)}$ in $L^{q*'}(R^N)$ as $n \to \infty$.

Proof : Since $u_n \to u$ in $L^{p^*}(R^N)$, there exist a subsequence still denoted (u_n) such that

$u_n(x) \to u(x)$ a.e .on R^N, $|u_n(x)| \le \eta(x)$ a.e. on R^N with $\eta \in L^{p^*}(R^N)$ (4.6)

Let $X_n = m(x)\dfrac{|u_n|^{p-2}u}{1+\epsilon^{1/p}|u|^{p-1}}$

Then $X_n(x) \to X(x) = m(x)\dfrac{|u(x)|^{p-2}u(x)}{1+\epsilon^{1/p}|u(x)|^{p-1}}$ a.e. on R^N,

$|X_n| \le ||m||_\infty |u_n||^{p-1} \le ||m||_\infty |\eta|^{p-1}$ in $L^{p^*}(R^N)$

Thus, from Lebesque's dominated convergence theorem one has

$X_n \to X = m(x)\dfrac{u^{p-2}u}{1+\epsilon^{1/p}|u|^{p-1}}$ in $L^{p^*}(R^N)$ as $n \to \infty$ so (1) is proved

Moreover, since $v_n \to v$ in $L^{q^*}(R^N)$ there exists a subsequences still denoted (v_n) such that

$v_n(x) \to v(x)$ a.e on (R^N), $v_n(x) \le \xi(x)$ a.e on R^N with $\xi \in L^{q^*}(R^N)$ (4.7)

Using **(B4),** one has $|Y_n| \le ||m_1||_\infty |(u_n, v_n)| \le \Gamma^{\alpha+\beta+2-p} ||m_1||_\infty |\eta|^{\alpha} |\xi|^{\beta+1}$ in $L^{p^*}(R^N)$ Let $Y_n = m_1(x)\dfrac{h(u_n, v_n)}{1+\epsilon\, h(u_n, v_n)}$, Then $Y_n(x) \to Y(x) = m_1(x)\dfrac{h(u(x), v(x))}{1+\epsilon\, h(u(x), v(x))}$ a.e. in R^N, as $n \to \infty$ So, we can apply the Lebesgue's dominated convergence theorem and then we obtain $Y_n(x) \to Y(x) = m_1(x)\dfrac{h(u(x), v(x))}{1+\epsilon\, h(u(x), v(x))}$ in $L^{p*'}(R^N)$ as $n \to \infty$

Remark 4.4 We can similarly prove that, as $n \to \infty$,

$n(x)|v_n|^{q-2}v_n (1+\epsilon^{1/q}|v_n|^{q-1})^{-1} \to n(x)|v|^{q-2}v(1+\epsilon^{1/q}|v|^{q-1})^{-1}$ in $L^{q^*} R^N$,

$n_1(x) k(u_n, v_n) (1+\epsilon |k(u_n, v_n)|)^{-1} \to n_1(x) k(u, v) (1+\epsilon|k(u, v)|)^{-1}$ in $L^{q^*} R^N$

To complete the continuity of T. Let us consider a sequence (u_n, v_n) such that $(u_n, v_n) \to (u, v)$ in $L^{p^*}(R^N)$, $\times$ $L^{q^*}(R^N)$, as $\to \infty$. We will prove that $(w_n, z_n) = T(u_n, v_n) \to (w, z) = T(u, v)$. Note that $(w_n, z_n) = T(u_n, v_n)$ if and only if

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$(-\Delta_p w_n + rm(x)|w_n|^{p-2} w_n) - (-\Delta_p w + rm(x)|w|^{p-2} w) = \hat{h}(x, u_n, v_n) - \hat{h}(x, u, v)$

$= (a+r)(X_n - X) + b(Y_n - Y)$

Multiplying by $(w_n - w)$ and integrating over R^N one has

$\int_{R^N}(|\nabla w_n|^{p-2} \nabla w_n - |\nabla w|^{p-2} \nabla w) \nabla (w_n - w) + r\int_{R^N} m(x)(|w_n|^{p-2} w_n - |w_n|^{p-2} w)(w_n - w)$

$= (a+r)\int_{R^N}(X_n - X)(w_n - w) + b\int_{R^N}(Y_n - Y)(w_n - w)$

$\leq (a+r)(\int_{R^N} m|X_n - X|^{p'})^{1/p'}(\int_{R^N}|w_n - w|^p)^{1/p} + b(\int_{R^N}|Y_n - Y|^{p'})^{1/p'}(\int_{R^N}|w_n - w|^p)^{1/p}$

Combining lemma 4.3 and the inequality

$$||x-y||^p \leq c\left((||x||^{p-2}x - ||y||^{p-2}y(x-y)\right)^{s/2}\left(||x||^p + ||y||^p\right)^{1-s/2} \quad (4.8)$$

We can conclude that $w_n \to w$ in $W_0^{1,p}(R^N)$ when $n \to \infty$.Similarly we show that $z_n \to z$ in $W_0^{1,q}(R^N)$ as $n \to \infty$ and then, the continuity of T is proved.

Compactness of the operator T. Suppose (u_n, v_n) a bounded sequence in $W_0^{1,p}(R^N) \times W_0^{1,q}(R^N)$ and let $(w_n, z_n) = T(u_n, v_n)$. Multiplying the first equality in the definition of T by w_n and integrating by parts on R^N, we notice the boundness of w_n in $W_0^{1,p}(R^N)$ and then we use the compact imbedding of $W_0^{1,p}(R^N)$ in $L^{p^*}R^N$, to conclude. The same argument is valid with (z_n) in in L^{q^*}, (R^N). Thus T is completely continuous. Since the set K is convex, bounded and closed in $L^{p^*}(R^N) \times L^{q^*}(R^N)$, the Schauder's fixed point theorem, yields existence of a fixed point for T and accordingly the existence of solution of system (4.2).

Proof of theorm 4.1 The proof will be given in three steps.

Step : 1 First to prove that (u_ϵ, v_ϵ) is bounded in $W_0^{1,p}(R^N) \times W_0^{1,q}(R^N)$. indeed assume that $||u_\epsilon|| \to \infty$ or $||v_\epsilon|| \to \infty$ as $\epsilon \to 0$. Let t_ϵ max $\{||u_\epsilon||; ||v_\epsilon||\}$,we $w_\epsilon = \frac{u_\epsilon}{t_\epsilon^{1/p}}$, $z_\epsilon = \frac{v_\epsilon}{t_\epsilon^{1/q}}$

We have $||w_\epsilon|| \leq 1$ and $||z_\epsilon|| \leq 1$ with either $||w_\epsilon|| \leq 1$ or $||z_\epsilon|| = 1$.

Dividing the first equation in (4.2) by $(t_\epsilon)^{1/q'}$ we obtain

$-\Delta_p w_\epsilon + rm(x)|w_\epsilon|^{p-2} w_\epsilon = (a+r)m(x)|w_\epsilon|^{p-2} w_\epsilon (1 + |\epsilon^{1/p} u_\epsilon|^{p-1})^{-1} + t_\epsilon^{-1/p'} bm_1(x) h(t_\epsilon^{1/p} w_\epsilon, t_\epsilon^{1/q} z_\epsilon)(1 + \epsilon|h(u_\epsilon, v_\epsilon)|)^{-1} + t_\epsilon^{-1/p'} f.$

Similarly dividing the second equation in (4.2) by $(t_\epsilon)^{1/q'}$ we obtain

$-\Delta_q z_\epsilon + rn(x)|z_\epsilon|^{q-2} w_\epsilon = (d+r)n(x)|w_\epsilon|^{\alpha} w_\epsilon (1 + |\epsilon^{1/p} u_\epsilon|^{\alpha+1})^{-1}$

$+ t_\epsilon^{-1/q'} cn_1(x) k(t_\epsilon^{1/p} w_\epsilon, t_\epsilon^{1/q} z_\epsilon)(1 + \epsilon|k(u_\epsilon, v_\epsilon)|)^{-1} + t_\epsilon^{-1/q} g.$

Testing the first equation in the above system by w_ϵ and using **(B4),** we obtain

$\int_{R^N}|\nabla w_\epsilon|^p \leq a\int_{R^N} m(x)|w_\epsilon|^p + b\Gamma^{-\alpha+\beta+2-p}\int_{R^N} m(x)^{\frac{\alpha+1}{p}}|w_\epsilon|^{\alpha+1} n(x)^{(\beta+1)/q}|z_\epsilon|^{\beta+1}$

$+ t_\epsilon^{-1/p'}\int_{R^N}|f||w_\epsilon|$., which, by the Holder inequality, implies

$\int_{R^N}|\nabla w_\epsilon|^p \leq a\int_{R^N} m|w_\epsilon|^p + b\Gamma^{-\alpha+\beta+2-p}(\int_{R^N} m|w_\epsilon|^p)^{\alpha+1/p}(\int_{R^N} n|w_\epsilon|^q)^{\beta+1/q}$

$+ t_\epsilon^{-1/p'}||f||_{(p^*)'}||z_\epsilon||_{p^*}$

Using the variational characterization of $\lambda_1(m, p)$ and the imbedding of $W_0^{1,p}(R^N)$ in $L^{p^*}(R^N)$. one has

$||w_\epsilon||^p \leq \frac{a}{\lambda_1(m,p)}||w_\epsilon||^p + b\Gamma^{-\alpha+\beta+2-p}\frac{||w_\epsilon||^{\alpha+1}}{\lambda_1(m,p)^{(\alpha+1)/p}} \, p\frac{||z_\epsilon||^{\beta+1}}{\lambda_1(n,p)^{(\beta+1)/q}}$

$+ c(p, N)\, t_\epsilon)^{-1/p'} \;||\; f \;||_{(p^*)'} \;||\; z_\epsilon \;||_{p^*}$, where $c(p, N)$ is the imbedding constant. $(\lambda_1 m,p)-$

$$a)\frac{(||w_\epsilon||^{p})^{(\beta+1)/q}}{\lambda_1(m,p)} \leq \frac{b\,\Gamma^{\alpha+\beta+2-p}(\;||z_\epsilon||^{q})^{(\beta+1)/q}}{\lambda_1(m,p)^{\frac{\alpha+1}{p}}\;\lambda_1(n,p)^{\frac{\beta+1}{q}}} + (t_\epsilon)^{-1/p'}\left(\int_{R^N}|f|^{p'}\right)^{\frac{1}{p'}}$$

$$\left(\int_{R^N}|\nabla w_\epsilon|^{p}\right)^{-\alpha/p} \qquad (4.9)$$

$(\lambda_1 \; m, \; p)-$

$$a)^{\alpha+1]/q}\;\frac{(\lim\sup||w_\epsilon||^{p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(m,p)^{\left(\frac{\alpha+1}{p}\right)}} \leq b^{\frac{\alpha+1}{p}}\;\frac{\Gamma^{\alpha+\beta+2-p\left(\frac{\alpha+1}{p}\right)}(\lim\sup||z_\epsilon||^{p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(m,p)^{\left(\frac{\alpha+1}{p}\right)^2}\;\lambda_1(n,p)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}} \qquad (4.10)$$

In a similar way, it can be obtain

$$(\lambda_1\, n, q)-d^{\frac{]\beta+1}{q}}\;\frac{(\lim\sup||z_\epsilon||^{p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(n,q)^{\frac{\beta+1}{q}}} \leq c^{\frac{\beta+1}{q}}\;\frac{\Gamma^{(\alpha+\beta+2-p)\frac{\alpha+1}{p}}(\lim\sup||w_\epsilon||^{p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(n,q)^{\left(\frac{\beta+1}{q}\right)^2}\;\lambda_1(m,p)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}} \qquad (4.11)$$

Multiplying term by term the expressions in (4.10) and (4.11), and using (3.8) we obtain

$$[(\lambda_1(m,p)-a)^{\frac{\alpha+1}{p}}(\lambda_1(n,q)-d)^{(\beta+1]/q} - b^{\frac{\alpha+1}{p}}c^{(\beta+1]/q}]\quad ò\frac{(\lim\sup||w_\epsilon||^{p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(m,p)^{\frac{\alpha+1}{p}}}\;\frac{(\lim\sup||z_\epsilon||^{p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(n,p)^{(\beta+1)/q}}$$

$\leq 0.$

Since conditions $(C_1)-(C_3)$ hold, one has $\lim\sup||w_\epsilon||^{p} = \lim\sup||z_\epsilon||^{p} = 0$.This yields a contradiction since $||w_\epsilon|| = 1$ or $||w_\epsilon|| = 1$, and consequently (u_ϵ, v_ϵ) is bounded in $W_0^{1,q}(\Omega)\times W_0^{1,q}(\Omega)$.

Step 2. $(\epsilon^{\frac{1}{p}} u_\epsilon;\ \epsilon^{\frac{1}{q}} v_\epsilon)$ converges strongly in $W_0^{1,p}(R^N)\times W_0^{1,q}(R^N)$. when ϵ approaches 0. It is obvious due to the boundness of (u_ϵ, v_ϵ) in $W_0^{1,p}(R^N)\times W_0^{1,q}(R^N)$).

Step 3. Now, from the strong convergence of (u_ϵ, v_ϵ) in $W_0^{1,p}(R^N)\times W_0^{1,q}(R^N)$ and a classical result in nonlinear analysis, we obtain

$-\Delta_p u_o + am(x)|u_0|^{p-2}u_0 + bm_1 m(x)\; h(u_0+v_0) + f$ in R^N

$-\Delta_q v_o = dn(x)|v_0|^{q-2}u_0 + cn_1 k(x)\; h(u_0+v_0) + g$ in R^N

$u_0(x) \to 0,\ v_0(x) \to 0,$ as $|x| \to +\infty$ .This completes the proof □

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