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RECURRENCE IN SPECIAL FINSLER SPACES

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Abstract
In the present paper we have introduced some special Finsler spaces like C- reducible, Semi C- reducible, C2- like, S3- like, S4- like, S^h - recurrent, T^h - recurrent and have derived results telling as to when a C- reducible Finsler space is C^h - recurrent, a C2- like Finsler space is C^h - recurrent, a semi C- reducible Finsler space is C^h - recurrent, a C^h - recurrent Finsler space is S^h - recurrent, a C- reducible Finsler space is S^h - recurrent, a S4- like Finsler space is S^h - recurrent and in the last we have also observed that a non-flat S3-like Finsler space is always S^h - recurrent.

1. Introduction

We consider an n-dimensional Finsler Space F_n referred to a local coordinate system x^i, where metric function *F(x, ẋ)* satisfies all the conditions usually imposed upon such a metric function. Now, we introduced some space special Finsler spaces which will form the subject matter of our studies.

A Finsler space F_n(n>2) with the non-zero length of torsion tensor C^i is called semi C-reducible [8] if

(1.1) $$C_{ijk} = \frac{p}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + \frac{q}{C^2}C_iC_jC_k,$$

where p and q = (1-p) do not vanish and h_{ij} is given by

(1.2) $$h_{ij} = g_{ij} - l_i l_j$$

and (1.3) $h^i_j = g^{ik}h_{jk}$.

The unit vector l^i in the direction of element of $\dot{x}^i$,is given by

(1.4) $$l^i = \frac{\dot{x}^i}{F(x,\dot{x})}$$

(1.5) $$F_i = F|_i = \frac{dF}{d\dot{x}^i} = l_i$$

and (1.6) $l_i = g_{ij}\, l^j$

A semi C-reducible space is of the first kind or of the second kind, according as p ≠ $\frac{n+1}{2}$ or p = $\frac{n+1}{2}$ [8].

A non-Riemannian Finsler space F_n is called C^h-recurrent [4], if the (h) hv- torsion tensor C_{ijk} satisfies

(1.7) $$C_{ijk|l} = k_l\, C_{ijk}$$

where, k_l is a covariant vector field.

The Berwald's v-connection is defined by

(1.8) $$G^i_{jk}(x,\dot{x}) = \partial^2 G^i / \partial\dot{x}^j \partial\dot{x}^k,$$

where (1.9) $2G^i(x,\dot{x}) = \gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k$.

This connection $G^i_{jk}$is not in general independent of directional arguments $\dot{x}^i$. A Finsler space in which G^i_{jk} is independent of directional arguments is called a Berwald's space. This space is characterised by the condition [5]

(1.10) $C_{ijk|l} = 0$.

It follows from (1.10) that each Berwald's space is a Landsberg space.

2. Properties of Recurrence

For carrying out further studies under this heading, we give the following definitions.

Definition (2.1) :

A tensor field T_{ij} is called h- recurrent, if

$$T_{ij|k} = \lambda_k T_{ij} ,$$

where λ_k is a covariant vector field.

Definition (2.2) :

A vector field T_i is called h- recurrent, if for a covariant vector field k_h, we have

$$T_{i|h} = k_h T_i \ .$$

On contracting (1.7) by g^{ik}, we get

$$C_{i|h} = k_h C_i \ .$$

Therefore, we can state:

Theorem (2.1)

In a C^h- recurrent Finsler space the covariant vector field C_i is h- recurrent.

In general, the condition (2.1) is only a necessary condition in a C^h- recurrent Finsler space. However, there are some special Finsler space where this condition is necessary as well as sufficient in order that the space may be C^h- recurrent. We discuss such special Finsler space as under.

In 1972, Matsumoto [6] introduced that notation of C- reducible Finsler space. A non-Riemannian Finsler space F_n(n>2) is called C- reducible if the (h) hv- torsion C_{ijk} of F_n is written in the form

(2.2) $C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)$,

where C_i is a torsion vector defined by $C_i = C^k_{ik}$. Taking the covariant derivative of (2.2) with respect to x^l, we get

(2.3) $C_{ijk|l} = \frac{1}{n+1}(h_{ij}C_{k|l} + h_{jk}C_{i|l} + h_{ki}C_{j|l})$.

At this stage if we assume that (2.1) holds, then equation (2.3) with the help of (2.2) gives that the space is C^h- recurrent. Therefore, in consequence of theorem (2.1), we have the following:

Theorem (2.2)

A necessary and sufficient condition in order that a C-reducible Finsler space is C^h- recurrent is that the vector field C_i is h- recurrent.

The notion of C-2 like Finsler space has been introduced by Matsumoto and Numata [7]. Such a Finsler space is characterized by a special form of torsion tensor C_{ijk} given by

(2.4) $C_{ijk} = \frac{1}{C^2} C_i C_j C_k$ with $C^2 \neq 0$.

The h-covariant derivative of (2.4) with respect to x^l gives

(2.5) $C^2 C_{ijk|l} + 2C_{ijk}C^p C_{p|l} = C_i C_j C_{k|l} + C_j C_k C_{i|l} + C_k C_i C_{j|l}$.

At this stage, we now consider $C_{k|l} = \psi_l C_k$, the equation (2.4) and (2.5) shows that C-2 like Finsler space is C^h- recurrent. Thus, in view of theorem (2.1), we can state:

Theorem (2.3)

A C-2 like Finsler space in C^h- recurrent. If the vector field C_i is h-recurrent.

Further, in the case of semi C-reducible Finsler space of second kind the equation (1.1) reduces to

(2.6) $C^2 C_{ijk} = \frac{C^2}{2} G_{(ijk)} \left(h_{ij} C_k\right) + \frac{1-n}{2} C_i C_j C_k$,

where, the symbol $G_{(ijk)}$ stands for the cyclic permutation of the indices i, j, k and summation thereafter. The h-covariant differentiation of (2.6) with respect to x^l gives

$$\text{(2.7)}\quad C^2 C_{ijk|l} + 2C_{ijk} C^p C_{p|l} = C^p C_{p|l} G_{(ijk)} \left(h_{ij} C_k\right) + \frac{C^2}{2} G_{(ijk)} \left(h_{ij} C_{k|l}\right) + + \frac{1-n}{2} G_{(ijk)} \left(C_i C_j C_{k|l}\right) .$$

If we now assume that $C_{i|l} = k_l C_i$, the equation (2.5) and (2.7) gives that the space under consideration is C^h-recurrent. Therefore, in virtue of theorem (2.1), we can state :

Theorem (2.4):

A semi C-reducible Finsler space of second kind is C^h- recurrent if and only if C_i is h-recurrent.

We now give the following definition which is similar to the definition of C^h- recurrent Finsler space as under:

Definition (2.3)

A non-Riemannian Finsler space is called S^h or T^h-recurrent according as the v-curvature tensor S_{hijk} or the T-tensor T_{gijk} of the space satisfies the relation

(2.8) $S_{hijk|l} = \psi_l S_{hijk}$,

or (2.8a) $P_{ijk|l} = \psi_l P_{ijk}$,

(2.9) $T_{hijk|l} = \psi_l T_{hijk}$,

respectively, where ψ_l is a covariant vector field.

It has been shows by Matsumoto [4] that if F_n is C^h- recurrent with recurrence vector k_l, then S_{hijk} satisfies

(2.10) $S_{hijk|l} = 2k_l S_{hijk}$.

This relation and (2.8) give

Theorem (2.5)

A C^h- recurrent Finsler space is S^h- recurrent.

In a C-reducible Finsler space [4] the v-curvature tensor S_{hijk} has been following form

(2.11) $S_{irlm} = (n+1)^{-2} \left(h_{im} C_{rl} + h_{rl} C_{im} - h_{il} C_{rm} - h_{rm} C_{il}\right)$,

with C_{ij} given by

(2.12) $C_{ij} = 2^{-1} C^2 h_{ij} + C_i C_j$.

On taking the h-covariant derivative of (2.11) and (2.12) with respect to x^p, we get

(2.13) $S_{ijkl|p} = (n+1)^{-2} \left(h_{il} C_{jk|p} + h_{jk} C_{il|p} - h_{ik} C_{jl|p} - h_{jl} C_{ik|p}\right)$

and (2.14) $C_{ij|p} = C^r C_{i|p} h_{ij} + C_{i|p} C_j + C_i C_{j|p}$,

Respectively.

At this stage, we now suppose that a C-reducible Finsler space is S^h- recurrent. Then from relations (2.11), (2.8) and (2.13), we have

(2.15) $h_{il}C_{jk|p}$+ $h_{jk}C_{il|p}$- $C_{il|p}$- h_{ik} - $h_{jl}C_{ik|p}$= $\psi_p(h_{il}C_{jk}+h_{jk}C_{il}$ - $h_{ik}C_{jl}$ - $h_{jl}C_{ik})$,

which after contraction with g^{il} yields

(2.16) (n-3) $C_{jk|p}+g^{il}C_{il|p}h_{jk}$ = ψ_p[(n-3) $C_{jk}+h_{jk}\, g^{il}C_{il}$]

We now contract (2.16) with g^{jk} and use (2.12) and (2.14) and get

(2.17) $C^r C_{r|p}$ = $\frac{c^2}{2}\psi_p$.

In view of (2.12), (2.14) and (2.17) , (2.16) reduces to

(2.18) $C_{i|p}$ C_j + $C_iC_{j|p}$ = ψ_p C_iC_j for n>3.

We now contract (2.18) with C^j which after making use of (2.17) gives

(2.19) $C_{i|p}$ = $\frac{1}{2}\psi_p C_i$,

where, we have taken into account the fact that C^2≠0. Conversely, the relations (2.11), (2.12), (2.13), (2.14) and (2.19) give the equation (2.8).

Therefore, we can state :

Theorem (2.6):

A C-reducible Finsler space F_n(n>3) is S^k- recurrent if and only if the vector field C^i is h-recurrent.

Theorems (2.2), (2.5) and (2.6) now yield the following :

Theorem (2.7)

The necessary and sufficient condition in order that a C-reducible Finsler space F_n (n>3) is S^k-recurrent is that it is C^k- recurrent.

From theorem (2.5), we observe that in general, S^h- recurrent is only a necessary condition for a Finsler space to be C^h- recurrent.

However, the theorem (2.7) shows that in a C-reducible Finsler space F_n(n>3) is S^h- recurrence is necessary as well as sufficient condition for the space to be C^h- recurrent. An S_3-like Finsler space is characterized by

(2.20) S_{ijkl} = $\frac{S}{(n-1)(n-2)}$ $(h_{ik}h_{jl}$ - $h_{il}h_{jk})$.

In such a Finsler space, the h-covariant derivative of (2.20) after using (1.2) and (1.6) gives

(2.21) $S_{ijkl|k}$ = $\frac{1}{(n-1)(n-2)}$ $S_{|k}(h_{ik}h_{jl}$ - $h_{il}h_{jk})$,

which in view of (2.20) reduces into the following form

(2.22) $S_{ijkl|k}$ = $\frac{S_{|k}}{S}$ S_{ijkl} ,

Provided that S≠0. Further from (2.20) and the fact that a Finsler space is called flat Finsler space if the v-curvature tensor S_{hijk} of the space satisfies

(2.23) S_{hijk}= 0.

Thus, it can easily be verified that an S3-like Finsler space is flat if and only if S≠0. Therefore, we can state :

Theorem (2.8)

A non-flat S3-lke Finsler space is S^k- recurrent.

We now consider an S4-like Finsler space which is characterized by

(2.24) $S_{hijk} = \frac{S}{(n-2)(n-3)}(h_{ij}h_{hk} - h_{ik}h_{hj}) + \frac{1}{(n-3)}(h_{hj}S_{ik}+h_{ik}S_{hj}- h_{hk}S_{ij}- h_{ij}S_{hk})$.

The h- covariant derivative of (2.24) with respect to x^l gives

(2.25) $S_{hijk|l} = \frac{S_{|l}}{(n-2)(n-3)}(h_{ij}h_{hk} - h_{ik}h_{hj}) + \frac{1}{(n-3)}(h_{hj}S_{ik|l}+h_{ik}S_{hj|l}-h_{hk}S_{hj|l}-h_{ij}S_{hk|l})$.

At this stage, we now suppose that

(2.26) $S_{ij|l} = k_l S_{ij}$,

For a covariant vector field k_l . The contraction of (2.26) with g^{ij} gives $S_{|l} = k_l$S and in view of (2.24) the relation (2.25) reduces to $S_{hijk|l} = k_l S_{hijk}$. Conversely, (2.26) is a necessary condition for Finsler space to be S^h- recurrent with recurrence vector field k_l.

Therefore, we can state:

Theorem (2.9)

For an S4-like Finsler space to be S^k- recurrent it is necessary and sufficient that the tensor field S_{ij} is h- recurrent.

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