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TWO SECTORS MODEL OF ENDOGENOUS GROWTH; THE UZAWA-LUCAS MODEL

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Introduction

In the mid 1980s a group of growth theorists led by Paul Romer (1986) became increasingly dissatisfied with exogenously driven explanation of long run productivity growth. This dissatisfaction motivated the construction of a class of growth models in which the key determinants of growth were endogenous to the model. The determination of long run growth within the model, rather than by some exogenously growing variables like unexplained technological progress is the desideratum for endogenous growth. Endogenous growth models deal with optimally selected amount of resources that are consumed and saved. it assumes a constant exogenous savings rate and a fixed level of technology. Two sector endogenous growth model distinguishes between physical and human capital. It deals with open economy in which physical and human capital are produced by identical production functions.

Conditions for Endogenous Growth

Constant return to scale operates in the sectors for goods and education. The Uzawa-Lucas model expressed in the form

$$y = C + \dot{K} + \delta K = AK^{\alpha} (uH)^{1-\alpha} \qquad \ldots (1)$$

and

$$\dot{H} + \delta H = \beta (1-u) H \qquad \ldots (2)$$

is the special case in which the education sector uses only human capital as an input i.e. $\eta = 0$. These production functions imply that diminishing returns do not arise when physical and human capital grow at the same rate. Thus in the steady state rates of returns remain constant and the economy can grow at a constant rate. According to Mulligan and Sala-i-Martin(1993). It is imperative to analyse whether more general specification of the production functions are consistent with positive growth in the steady state i.e. with endogenous growth.

Modifying equations (1) and (2) to

$$Y = C + \dot{K} + \delta K = A (VK)^{\alpha} (uH)^{\alpha_2} \qquad \ldots (3)$$

and

$$\dot{H} + \delta H = B [(1-v)K]^{\eta 1} . [(1-u)H]^{\eta 2} \qquad \ldots (4)$$

Thus Cobb-Douglas forms of the production function is retained subject to $\alpha_1 + \alpha_2$ and $\eta_1 + \eta_2$ depart from unity so that constant returns to scale need not apply. If a sector exhibits diminishing returns i.e. $\alpha_1 + \alpha_2 < 1$ then the usual competitive framework is adhered to. If this factor has an exponent of $1 - \alpha_1 - \alpha_2$ then constant returns again apply at the level of an individual producer. The relevant consideration is that diminishing returns $\alpha_1 + \alpha_2 > 1$ apply to the factors that can be accumulated. The model can also have increasing returns i.e $\alpha_1 + \alpha_2 > 1$ within a competitive setup. For example for the production of y an individual firms inputs of K and H having exponents α_1 and $(1 - \alpha_1)$ respectively so that constant returns apply for an individual firm. The economy' aggregate of H could appear as an additional input in the production function with an exponent of $\alpha_1 + \alpha_2 - 1$ where $\alpha_2 > 1 - \alpha_1$ The key consideration is that increasing returns $\alpha_1 + \alpha_2 > 1$ apply to the factors that can be accumulated by the overall economy.

The Uzawa-Lucas Model

The Basic Frame Work

This model focuses on production of human capital i.e. $\eta = 0$. This setting is the extreme case in which the education sector is relatively intensive in human capital $(\eta \leq \alpha)$. This model assumes relative factor intensities and non-negativity constraints on gross investment in **K** and **H** are not binding. The specification $\eta = 0$ implies $v = 1$ i.e since **K** is not productive in the education sector all of it is used in the goods sector. The production functions

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$$Y = C + \dot{K} + \delta K = A\,(VK)^{\alpha}\,(uH)^{1-\alpha} \quad \ldots (5)$$

and

$$\dot{H} + \delta H = B\,[(1-v)K]^{\eta}\,.\,[(1-u)H]^{1-\eta} \quad \ldots (6)$$

Can be simplified to

$$Y = C + \dot{K} + \delta K = AK^{\alpha}(uH)^{1-\alpha} \quad \ldots (7)$$

and

$$\dot{H} + \delta H = B\,(1-u)\,H \quad \ldots (8)$$

It is relevant to express the system in terms of variables that will be constant in steady state. A specification that facilitates the dynamic analysis involves the ratio $\omega = K/H$ and $\chi = C/K$. Using these definitions along with eq (7) and (8) the growth rates of K and H can be obtained as

$$\Upsilon_K = A\,u^{(1-\alpha)}\,.\,\omega^{-(1-\alpha)} - \chi - \delta, \quad \ldots (9)$$

$$\Upsilon_H = B.\,(1-u) - \delta \quad \ldots (10)$$

Hence the growth rate of ω is given by

$$\Upsilon_\omega = \Upsilon_K - \Upsilon_H = A.u^{(1-\alpha)}\,\omega^{-(1-\alpha)} - B.\,(1-u) - \chi \quad \ldots (11)$$

The first order conditions can be used to show that the growth rate of consumption is given by the familiar formula $\Upsilon_c\ (1/\theta)(r-\rho)$ where r equals the net marginal product of physical capital in the production of goods, $\alpha A.\ u^{(1-\alpha)}\ .\ \omega^{-(1-\alpha)} - \delta.$ Thus the growth rate of consumption is given by

$$\Upsilon_c = (1/\theta)[\alpha A.\ u^{(1-\alpha)}\,.\,\omega^{-(1-\alpha)} - \delta - \rho] \quad \ldots (12)$$

The growth rate of x follows from eq (12) and eq (9) as

$$\Upsilon_x = \Upsilon_c - \Upsilon_k = \frac{\alpha-\theta}{\theta}\,.\,A\,u^{(1-\alpha)}\,\omega^{(1-\alpha)} + \chi - \left(\frac{1}{\theta}\right)[\delta\,.(1-\theta) + \rho] \quad \ldots (13)$$

Solution of the Uzawa-Lucas Model
The Hamiltonian expression for this model can be expressed as

$$J = u\,.\,(c).\ e^{-\rho t} + V.\,(AK^{\alpha}\,.\,(u\,H)^{1-\alpha} - c - \delta K] + \mu\,.\,[B.\,(1-u).H - \delta H] \quad \ldots (14)$$

The term in the first set of brackets equals K and the term in the second set of brackets equals H. Defining $\omega = K/H$ and $\chi = C/K$, then the growth rates of K and H are given by

$$\Upsilon_K = A\,u^{1-\alpha}\,\omega^{-(1-\alpha)} - \chi - \delta, \quad \ldots (15)$$

$$\Upsilon_H = B\,(1\text{-}u) - \delta_i \qquad \ldots (16)$$

The growth rate of **ω** is given by

$$\Upsilon_\omega = \Upsilon_K - \Upsilon_H = A\,u^{1-\alpha}\,\omega^{-(1-\alpha)} - \chi - B.\,(1\text{-}u) \qquad \ldots (17)$$

The first order conditions

$$\frac{\delta J}{\delta C} = 0 \text{ and } \frac{\delta J}{\delta U} = 0 \text{ lead respectively to } u'(c) = V.e^{\rho t} \quad \ldots (18)$$

$$\frac{\mu}{v} = (A/B)\,.\,(1\text{-}\alpha)\,u^{-\alpha}\,\omega^{\alpha} \qquad \ldots (19)$$

The condition $\dot{v} = -\,\delta J/\delta K$ implies

$$\frac{\dot{v}}{v} = -A\alpha\,.\,u^{1-\alpha}\,\omega^{-(1-\alpha)} + \delta_i \qquad \ldots (20)$$

The condition $\dot{\mu} = -\,\delta J/\delta H$ implies

$$\frac{\dot{\mu}}{\mu} = -(v/\mu).\,A.\,(1\text{-}\alpha)\,u^{1-\alpha}\,\omega^{\alpha} - B.\,(1\text{-}u) + \delta$$

Substituting for **v/μ** from eq (19) and simplifying the result as

$$\frac{\dot{\mu}}{\mu} = -\,\mathbf{B} + \boldsymbol{\delta}_i \qquad \ldots (21)$$

Differentiating eq (18) with respect to time and using $\mathbf{u(c) - c^{1-\theta} - 1/(1-\theta)}$ and the expression for $\frac{\dot{v}}{v}$ in eq (20) to get the usual equation for consumption growth

$$\Upsilon_c = (1/\theta)\,.\,[A\,\alpha\,.\,u^{1-\alpha}\,\omega^{-(1-\alpha)} - \delta - \rho] \qquad \ldots (22)$$

The result corresponds to eq (12). The growth rate of $\boldsymbol{\chi}$ can be determined from eq (22) and eq (15) to get the formula as mentioned in eq (13) i.e.

$$\Upsilon_\chi = \Upsilon_c - \Upsilon_K = \frac{\alpha - \theta}{\theta}\,.\,A\,u^{1-\alpha}\,\omega^{-(1-\alpha)} + \chi - (1/\theta)\,[\,\delta.\,(1-\theta) + \rho] \qquad \ldots (23)$$

Differentiating eq (19) with respect to time and using the formula for $\frac{\dot{v}}{v}$ from eq (20), $\frac{\dot{\mu}}{\mu}$ from eq(21) and $\boldsymbol{\Upsilon_\omega}$ from eq (17) we get after simplifying

$$\Upsilon_u = B.\,(1\text{-}\,\alpha)/\alpha + Bu - \chi \qquad \ldots (24)$$

The result appears in eq (14). Equation (17), eq (23) and eq (24) from a system of three differential equations in the variables ω, χ and μ where the state variable ω begins at some value $\omega(0)$

The steady state of this system can be found readily by setting the three time derivatives to zero. Defining the combination of parameters as

$$\psi = [\rho + \delta . (1-Q)]/ BQ$$

Then the results are

$$\overset{*}{\omega} = \left(\frac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}} [\psi + (\theta - 1)/\theta)]$$

$$\overset{*}{\chi} = B. (\psi + 1/\alpha - 1/\theta)$$

$$\overset{*}{u} = \psi + (\theta - 1)/\theta \qquad \dots (25)$$

These values are given in eq (16). The steady state rate of return which equals the net marginal product of K in the goods sector and the net marginal product of **H** in the education sector is

$$\overset{*}{r} = B - \delta$$

The corresponding steady state growth rate of **y, c, K** and **H** is

$$\overset{*}{\gamma} = (1/\theta) . (B - \delta - \rho)$$

The values for $\overset{*}{r}$ and $\overset{*}{¥}$ are shown in eq. 17

Defining z to be the gross average product of physical capital

$$z = A\, u^{1-\alpha}\, \omega^{-(1-\alpha)}$$

The steady state value of z can be determined from eq (25) to be $\overset{*}{z} = B/\alpha$

The system of three differential equations as expressed in eq(17), eq(23) and eq(24) can be restated as

$$\Upsilon_{\omega} = (z - \overset{*}{z}) - (\chi - \overset{*}{\chi}) + B.(u - \overset{*}{u}) \qquad \dots (26)$$

$$\Upsilon_{\chi} = \frac{\alpha - \theta}{\theta}(z - \overset{*}{z}) + (\chi - \overset{*}{\chi}) \qquad \dots (27)$$

$$\Upsilon_{u} = B. (u - \overset{*}{u}) + (\chi - \overset{*}{\chi}) \qquad \dots (28)$$

The definition of z implies

$$\Upsilon_{z} = (1- \alpha) (\Upsilon_{u} - \Upsilon_{\omega}) = - (1 - \alpha) (z - z) \qquad \dots (29)$$

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The results for Υ_z, Υ_χ and Υ_u are in equation

$$\Upsilon_z = - (1 - \alpha)(z - \overset{*}{z}) \quad \text{... (30)}$$

$$\Upsilon_\chi = \frac{\alpha - \theta}{\theta}(z - \overset{*}{z}) + (\chi - \overset{*}{\chi}) \quad \text{... (31)}$$

$$\Upsilon_u = B.(u - \overset{*}{u}) + (\chi - \overset{*}{\chi}) \quad \text{... (32)}$$

Where is **z** the steady state value of **z**. Eq (16) i.e

$$\overset{*}{\omega} = (\alpha A/B)^{1/(1-\alpha)} . [\psi + (\theta - 1)/\theta]$$

$$\overset{*}{\chi} = B$$

$$u = \psi + (\theta - 1)/\theta$$

Equation (29) can be integrated to get eq

$$\frac{z - \overset{*}{z}}{z} = \left[\frac{z(0) - \overset{*}{z}}{z(0)}\right] . e^{-(1-\alpha)^{*} zt}$$

Where **z(0)** is the initial value of **z**. This eq. can be restated to solve for **z** as

$$z = \frac{\overset{*}{z} . z(0)}{\{\overset{*\bullet}{z} . e^{-(1-\alpha)\overset{*\bullet}{z} t} + z(0)[1 - e^{-(1-\alpha)\overset{*\bullet}{z} t}]\}} \quad \text{... (33)}$$

The above equation implies $\overset{*\bullet}{z}$ as t $\to$ ∞. If z(0) > $\overset{*\bullet}{z}$ then $\overset{*\bullet}{z}$ <0 and z > $\overset{*\bullet}{z}$ from all t, whereas if z(0) < $\overset{*\bullet}{z}$ then $\overset{*\bullet}{z}$ > 0 and z< $\overset{*\bullet}{z}$ for all t.

The characteristics of the stable path of x and u i.e. the path along which ¡ = ¡ $^{*\bullet}$ and $\overset{*\bullet}{u}$ assuming z(0) > $\overset{*\bullet}{z}$. So that z - $\overset{*\bullet}{z}$ declines monotonically over time. Eq 27 can be restated as

$$\Upsilon_\chi = \chi - \overset{*\bullet}{\chi} + \frac{\alpha - \theta}{\theta}\Omega(t) \quad \text{... (34)}$$

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Where **Ω(t)**= $z - \overset{*\bullet}{z}$ is a monotonically decreasing function of time. If~<¯then the term on the right of eq (34) is negative but declining in magnitude overtime. If ¡⁼á $\overset{*\bullet}{¡}$ for some finite t, then the equation implies $\overset{*\bullet}{¡}$ < 0 for all subsequent t. Since the magnitude of ¡⁼asymptotically exceed some finite time. The stable path therefore features ¡ > $\overset{*\bullet}{¡}$ for all t.

If $\overset{*\bullet}{¡}$ ñ for some t then eq (34) implies $\overset{*\bullet}{¡}$ >0 for all subsequent t because the negative term on the right decreases in size overtime. Hence ¡⁼would diverge form $\overset{*\bullet}{¡}$ and approach ∞. The stable path therefore involves $\overset{*\bullet}{¡}$ < 0 for all t.

Assumingα>Θ or z(0) < $\overset{*\bullet}{z}$ the columns for ¡⁼J $\overset{*\bullet}{¡}$ and ¡⁼in the below table No. 1 epitomises the results eq(28) determines the behaviour of u, given the behaviour of ¡. Suppose z(0)> $\overset{*\bullet}{z}$ and α<Θ , so that ¡ > $\overset{*\bullet}{¡}$ and $\overset{*\bullet}{¡}$ <0. If u≤ $\overset{*\bullet}{u}$ for some t, then eq (28) implies $\overset{*\bullet}{u}$ > 0 for all subsequent t because the term -(¡⁼J $\overset{*\bullet}{¡}$ F⁼in eq (28) is negative and decreasing in size overtime. Therefore $\overset{*\bullet}{u}$ < 0 holds for t. The behaviour of u - $\overset{*\bullet}{u}$ and $\overset{*\bullet}{u}$ are shown for the various sign combination of z(o) - $\overset{*\bullet}{z}$ and α - Θ in table no. 1.

Table No 1: Transitional Behaviour of and u

z (0) - $\overset{*\bullet}{z}$	~ - **Θ**	¡⁼J $\overset{*\bullet}{¡}$	¡	u - $\overset{*\bullet}{u}$	u
> 0	< 0	> 0	< 0	> 0	< 0
> 0	> 0	< 0	> 0	< 0	> 0
= 0	- -	= 0	= 0	= 0	= 0
<0	< 0	< 0	> 0	< 0	> 0
- -	= 0	= 0	= 0	= 0	= 0

It is relevant to examine how the starting value **z(o) -** $\overset{*\bullet}{z}$ relates to the starting value of the state variable ω. Using eq(27) to substitute for ¡⁼J $\overset{*\bullet}{¡}$ ⁼in the formula for $ß_{µ⁼}$in eq (26) then we get

$$\Upsilon_{\omega} = (\alpha/\theta)\,(z - \overset{\bullet}{z}) - \Upsilon_{\chi} + B.\,(u - \overset{*\bullet}{u}) \qquad \dots (35)$$

Suppose α≤0 and z(0) > $\overset{*\bullet}{z}$ **.** In this case the conditions **(z -** $\overset{*\bullet}{z}$ **)> 0,** $\overset{*\bullet}{¡}$ **á0** and u - $\overset{*\bullet}{u}$ ≥ 0 which imply $\Upsilon_{\omega} > 0$ in eq(35). Hence the system can be on the stable path only if ω(0)> $\overset{*}{\mu}$. More over ω then rises monotonically from ω**(0) t**owards $\overset{*}{\mu}$ because $\Upsilon_{\omega} > 0$. Hence the monotonic decline in Z corresponds to a monotonic rise in $\overset{*}{\mu}$.

This result implies that a lower starting value of the state variable ω(0) is associated with a higher initial value z(0). By similar reasoning **z(0)** < $\overset{*\bullet}{z}$ corresponds to ω**(0)** > $\overset{*}{\mu}$ and **z(0)** = $\overset{*\bullet}{z}$ = $\overset{*\bullet}{z}$ **to** ω**(0) = 0 =** $\overset{*}{\mu}$ **.**

To deal with the case in which α>θ, substitute for u - $\overset{*\bullet}{u}$ from eq(28) in to eq. (26) to get

$$\Upsilon_{\omega} = (z - \overset{*}{z}) + \Upsilon_{u} \qquad \text{... (36)}$$

We can use this equation when $\alpha > \theta$ to show that $z(0) > \overset{*}{z}$ ($z[0] < \overset{*}{z}$) corresponds to $\omega(0) < \overset{*}{\mu}$ ($\omega(0) > \overset{*}{\mu}$)

It can be concluded that $z(0) >=< \overset{*}{z}$ corresponds to $\omega(0) <=> \overset{*}{\mu}$ for all configuration of α and θ. Moreover a smaller $\omega(0)$ matches up with a higher $z(0)$. Thus z is high or low initially depends only on whether physical capital is scarce or abundant relative to human capital

The rate of return, r equals the net marginal product of physical capital in the production of goods which equals $\alpha z - \delta$. Therefore r moves together with **z** and inversely with ω. Eq(22) implies that the growth rate of **c** is given by

$$\Upsilon_{c} = (1/\theta)\,(\alpha z - \delta - \rho) \qquad \text{... (37)}$$

Since Υ_c moves directly with z, it moves inversely with ω. The growth rate of K is given by

$$\Upsilon_{K} = \Upsilon_{c} - \Upsilon_{\chi} = (1/\theta)\,(\alpha z - \delta - \rho) - \Upsilon_{\chi}$$

Where we substituted for Υ_c from eq(37). If we substitute for Υ_χ from eq(33) and use the formula $\overset{*}{z} = (B/\alpha)$ and

$$\overset{*}{\gamma} = (1/\theta)\,(B - \delta - \rho)$$

then we get

$$\Upsilon_{K} = (z - \overset{*}{z}) - (\chi - \overset{*}{\chi}) \qquad \text{... (38)}$$

the formula that appears in eq

$$\Upsilon_{K} = (z - \overset{*}{z}) - (\chi - \overset{*}{\chi})$$

The growth rate of H is given by

$$\Upsilon_{H} = \Upsilon_{K} - \Upsilon_{\omega}$$

If we substitute for Υ_K from eq (15) Υ_ω from eq(37) then we can simplify to get

$$\Upsilon_{H} = \overset{*}{\gamma} - B\,(u - \overset{*}{u}) \qquad \text{... (39)}$$

The formula that appears in eq

$$\Upsilon_{H} = \overset{*}{\gamma} - B\,(u - \overset{*}{u})$$

Since $\Upsilon = AK^{\alpha}.(uH)^{1-\alpha}$ the growth rate of output is given by

$$\Upsilon_{H} = \alpha\Upsilon_{K} + (1 - \alpha).(\Upsilon_{u} + \Upsilon_{H})$$

If we substitute for Υ_κ from eq(38) for Υ_υ from (28) and for Υ_H from eq (39) we get

$$\Upsilon_y = \dot{\gamma} + \alpha.(z - \dot{z}) - (\chi - \dot{\chi}) \quad \ldots (40)$$

The formula that appears in

$$\Upsilon_y = \dot{\gamma} + \alpha.(z - \dot{z}) - (\chi - \dot{\chi})$$

Broad output is given by

$$Q = Y + (\mu/v).\ B.\ (1\text{-}u)\ H = AK^{\alpha}.\ (uH)^{1-\alpha} + (\mu/v).\ B.\ (1 - u)\ H$$

Where μ/v, the shadow price of human capital in units of goods is given in the above eq. If we substitute out for μ/v then we get

$$Q = y.\ (1-\alpha+\alpha u)/u$$

Hence the growth rate of broad output is given by

$$\Upsilon_Q = \Upsilon_y - \Upsilon_u\ .\ (1-\alpha)/\ (1-\alpha+\alpha u) \quad \ldots (41)$$

The Generalised Uzawa-Lucas Model

The generalised form of the Uzawa-Lucas model maintains the assumption that education is relatively intensive in human capital $n < \alpha$ but allows for the presence of physical capital in the education sector n>0. It is observed from the equation

$$\rho = \mu/v = (A/B)\ (\alpha/\eta)^{\eta}\ \ [(1-\alpha)/\ 1-\eta]^{1-\eta}\ .\ (vK/u\ H)^{\alpha-\eta}$$

and the equation

$$\Upsilon_p = A\ \theta^{\alpha/\eta - \alpha}\ .\ [\alpha\ \theta^{1/(\alpha-\eta)}\ .\ \rho^{1-\alpha/(\eta-\alpha)}\ \rho^{\eta/(\alpha-\eta)}]^{-(1-\alpha)}$$

for the case $\eta<\alpha$ that VK/uH - the ratio of physical capital employed in production to human capital in production = converges monotonically to its steady state value. This result implies that the rate of return, r and the growth rate of consumption, Υ_c converge monotonically to their steady state values. These results are the same as those for the uzawa - Lucas case $\eta = 0$.

We have carried out simulations in which α=is set at 0.4 and the parameter η=is varied between 0 and 0.4. Assuming familiar values for the other parameters: a representative case is δ== 0.05, P = 0.02, η = 0.01 and θ=3. For η = 0 we set B = 0.13, so that steady state interest rate is 0.08 and the steady state per capita growth rate is 0.02.

As η approaches α the simulations show that the policy functions for u and χcontinue to be monotonically and inversely related to ω. The numerical results suggest that higher values of η=tend to make Υ_y slope upward in the vicinity of the steady state. If we assume reasonable value of the underlying parameters then the main qualitative results of the Uzawa - Lucas model are likely to be maintained when we drop the unrealistic assumption that the education sector has no inputs of physical capital (η=0). Another difference in the generalised model is that the range in which the inequality restriction $u \leq 1$and $\overset{*}{K}$ + δ K $\geq$0 are not binding narrows as η rises towards α. This result makes sense that this range compresses to 0 when $\eta = \alpha$. Having the reasonable assumption that η is much less than α- even if η is now positive - then we still find there exists a broad range of values of ω around the steady for which the inequality constraints are not binding.

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The Model with Reversed Factor Intensities

The environment in which the education sector is relatively intensive in human capital corresponds to $\alpha > \eta \geq 0$. The implication of reversed factor intensities relate to $\alpha < \eta$. The assumption that education is relatively intensive in physical capital is implausible. If we interpret K and H not as physical and human capital but in some alternate way then the reversed factor intensities might apply. The condition $\alpha > \eta$ implies that equation

$$\Upsilon_{\rho} = A\phi^{\alpha} / (\eta - \alpha) . [\alpha \phi^{1/\alpha - \eta} . P^{(1-\alpha)/\eta - \alpha} - (1 - \alpha) P^{\eta/\alpha - \eta}]$$

is an unstable differential equation in the variable $\rho = \mu/v$ Any departure of p from its steady state value would be magnified over time. This unstable behaviour would then be transmitted to the ratio vK/uH from the equation .

$$\rho = \mu/v = (A/B).(\alpha/\eta)^{\eta} [(1-\alpha)/(1-\eta)]^{1-\eta} (v\kappa/\mu H)^{\alpha-\eta}$$

The ratio determines the marginal product of physical capital in the production of goods and hence determines r and Υ_c. The unstable behaviour of vK/uH would be transmitted accordingly to r and Υ_c.

Transitional Dynamics

The dynamic system for ω, χ and μ consists of equations

$$\Upsilon^{*}_{\omega} = \Upsilon\kappa - \Upsilon H = A.\ \mu^{(1-\alpha)} \omega^{-(1-\alpha)} - B.\ (1-u) - \chi$$

$$\Upsilon_{\chi} = \Upsilon_{c} - \Upsilon_{\kappa} = \left(\frac{\alpha - \theta}{\theta}\right) .A.\ \mu^{(1-\alpha)} \omega^{-(1-\alpha)} + \chi - (1/\theta).[\delta.\ (1-\theta) + \rho]$$

$$\Upsilon_{u} = B.\ (1-\alpha)/\alpha + Bu - \chi$$

It is convenient to work with a transformed system that replaces ω by the gross average product of physical capital in the production of goods denoted by **Z**

$$Z = Au^{(1-\alpha)} . \omega^{-(1-\alpha)}$$

The gross marginal product of physical capital equals αz and the rate of return is $r = \alpha z - \delta$. Although the variable **z** is a combination of a state variable ω and a control variable u in the equilibrium, z relates in a simple way to ω .We can determine the initial value Z (0) from the initial value $\omega(0)$. The system given by the above equations can be restated in terms of z, χ and u as

$$\Upsilon_{z} = -(1-\alpha).(z - z^{*})$$

$$\Upsilon_{\chi} = \left(\frac{\alpha - \theta}{\theta}\right).(z - z^{*}) + (\chi - \chi^{*})$$

Where $\mathbf{z}^{*}$ is the steady state value of **z.** Equations

$$\omega^{*} = (\alpha\ A/B)^{1/1-\alpha} . [\psi + {}^{(\theta-1)/\theta}]$$

$$\chi^{*} = B(\psi + 1/\alpha - 1/\theta)$$

and definition of **z** in equation

$$z = Au^{(1-\alpha)}\omega^{-(1-\alpha)}$$

imply that this steady value is given by

$$z^{*} = B/\alpha$$

Conclusion

The Uzawa - Lucas model provides a perspective on the effects of imbalance between K and H that differs from that of one sector model Υ_c is always inversely related to ω. The growth rates tend to rise with the amount of the imbalance between human and physical capital if human capital is abundant relative to physical capital $(\omega < \omega^{*})$ but they tend to fall with the amount of imbalance if human capital is relatively scarce $(\omega > \omega^{*})$. The model predicts accordingly that an economy would recover faster in response to a war that destroyed mainly physical capital than to an epidemic that destroyed mainly human capital. The underlying source of the new results is the assumption that the education sector is relatively intensive in human capital. If $\omega > \omega^{*}$ e.g. then the marginal product of human capital in the goods sector is high and the growth would be expected to occur mainly because of the high growth rate of human capital. The high level of ω implies a high wage rate and therefore a high cost of operation for the sector, education which is relatively intensive in human capital. This effect motivates people to allocate human capital to production of goods, rather than to education, the sector that produces the relatively scarce factor H. This effect tends accordingly to retard the economy's growth rate when ω rises above ω^{*}.

AK model can be extended to allow for two sectors, one that produced consumables **C** and physical capital **K** and another that created human capital **H**. If the sectors have the same factor intensities, then the main new results about growth come from the restriction that gross investment in each type of capital good must be non negative. This restriction generates an imbalance effect where by the growth rate of output rises with the magnitude of the gap between the ratio K/H and its steady state value. The assumption of equal factor intensities neglects a key aspect of education, it relies heavily on educated people as an input. The structure is modified to specify that the production of human capital is relatively intensive in human capital. This change in specification alters the conclusion about the imbalance effect. The growth rate tend to rise with the extent of the imbalance if human capital is relatively abundant but to decline with the extent of the imbalance if human capital is relatively scarce. The results imply that an economy would recover rapidly in reaction to a war that destroyed primarily physical capital but would rebound only slowly from an epidemic that eliminated mainly human capital.

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